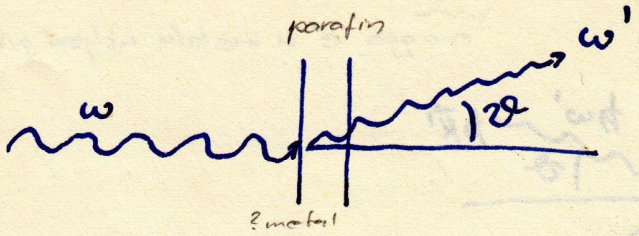
Vježbe iz

UAM-a II

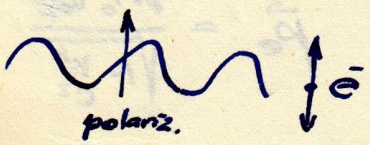
(1-22)

Uvod u Kvantnu mehaniku

Compton-ov efekt (1922.)



1° $\omega \to \omega'$ neobjašnjivo u klas. el.dinamici -sujetlost ne može promijeniti frekv.



elektron zatitra u vanjskom polju istom frekvencijom

$$m\ddot{\vec{r}} = e\vec{E}_0 e^{-i\omega t}$$

- Konvencija da se kasnije uzima Re dio

$$m\ddot{x} = eE_0 e^{-i\omega t} \Rightarrow m\dot{x} = \frac{eE_0}{-i\omega} e^{-i\omega t} \Rightarrow mx = \frac{eE_0}{-\omega^2} e^{-i\omega t} \Rightarrow x = -\frac{eE_0}{m\omega^2} e^{-i\omega t}$$

$$m\ddot{y} = 0 \Rightarrow \dot{y} = C_y \Rightarrow y = C_y t$$

$$m\ddot{z} = 0 \Rightarrow \dot{z} = C_z \Rightarrow z = C_z t$$

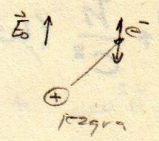
$$\Rightarrow \vec{r} = -\frac{e\vec{E}_0}{m\omega^2} e^{-i\omega t}$$

$$\vec{r} = \vec{A} e^{-i\omega t} \Rightarrow -m\omega^2 \vec{A} = e\vec{E}_0 \Rightarrow \vec{A} = -\frac{e\vec{E}_0}{m\omega^2}$$

$$\Rightarrow \vec{r} = -\frac{e\vec{E}_0}{m\omega^2} e^{-i\omega t}$$

dipolni moment se mijenja u vremenu

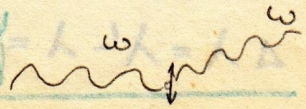
tj. dipolni moment



$$\vec{P}(t) = e\cdot\vec{r} = -\frac{e^2\vec{E}_0}{m\omega^2} e^{-i\omega t}$$

$\Rightarrow$ mijenja se u vremenu, pa $\bar{e}$ zrači energiju tj. el. polje <u>iste frekv.</u>

$$\vec{E}_{dip.mom.} \propto \ddot{\vec{P}} = \frac{e^2\vec{E}_0}{m} e^{-i\omega t}$$



$$\left.\begin{array}{l} E = mc^2 \\ E = \hbar\omega \end{array}\right\} \Rightarrow \quad p = mc = \frac{\hbar\omega}{c} = \frac{h\cdot\nu}{c} = \frac{h}{\lambda} = \frac{h}{2\pi}\left(\frac{2\pi}{\lambda}\right)^{K}$$

$\vec{K} = \frac{2\pi}{\lambda}\cdot\hat{n}$ – u smjeru el. magn. zrač.

2° Einstein :

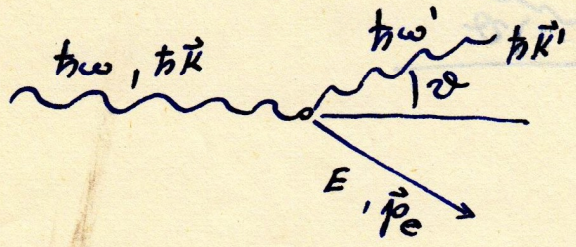
foton $\quad E_f = \hbar\omega \qquad \vec{p}_f = \hbar\vec{K} = \hbar\frac{2\pi}{\lambda}\hat{n}$

$$= \hbar\frac{\omega}{c}\hat{n}$$

pojednostavljenje:

X zrake $\quad \lambda \approx 1\,\text{Å} \Rightarrow E = \hbar\omega \gg 10\,\text{eV}$

energija e^- u metalu uslijed gibanja $\Big\}$ e^- miruju aprox.



i) $\hbar\vec{K} = \hbar\vec{K}' + \vec{p}_e$

ii) $\hbar\omega + \underset{V=0}{m_0c^2} = \hbar\omega' + \frac{m_0c^2}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \vec{p}_e = \frac{m_0\vec{V}_e}{\sqrt{1-\frac{V^2}{c^2}}}$

i) $\hbar^2\left(K^2 + K'^2 - 2\underbrace{\vec{K}\cdot\vec{K}'}_{K\cdot K'\cos\vartheta}\right) = p_e^2 = \frac{m_0^2V^2}{1-\frac{V^2}{c^2}} \qquad \hbar\vec{K} = \hbar\frac{\omega}{c}\cdot\hat{n}$

ii) $\hbar^2(\omega^2 - \omega'^2) + 2\hbar m_0c^2(\omega-\omega') + m_0^2c^4 = \frac{m_0^2c^4}{1-\frac{V^2}{c^2}}$

$m_0^2c^2\frac{1+\frac{V^2}{c^2}}{1-\frac{V^2}{c^2}}$

$$\frac{\hbar^2}{c^2}(\omega^2+\omega'^2) + 2\hbar m_0(\omega-\omega') = m_0^2c^2\left(\frac{1}{1-\frac{V^2}{c^2}} - 1\right) = \frac{m_0^2V^2}{1-\frac{V^2}{c^2}}$$

$$+2\frac{\hbar^2}{c^2}\omega\omega' + \frac{\hbar^2}{c^2}(\cancel{\omega^2}+\omega'^2) + 2\hbar m_0(\omega-\omega') = \hbar^2\left(\frac{\cancel{\omega^2}}{c^2} + \frac{\omega'^2}{c^2} - 2\frac{\omega\omega'}{c^2}\cos\vartheta\right)$$

$$2\hbar^2\omega\omega' + \cancel{\hbar^2\omega'^2} + 2\hbar m_0c^2(\omega-\omega') = \cancel{\hbar^2\omega'^2} - 2\hbar^2\omega\omega'\cos\vartheta$$

$$2\hbar^2\omega\omega' - 2\hbar m_0c^2(\omega-\omega') - 2\hbar^2\omega\omega'\cos\vartheta = 0$$

$$\hbar^2\omega\omega'(1-\cos\vartheta) = \hbar m_0c^2(\omega-\omega')$$

$\cos\vartheta = 1 - 2\sin^2\frac{\vartheta}{2}$

$1-\cos\vartheta = 2\sin^2\frac{\vartheta}{2}$

$\omega = \frac{2\pi c}{\lambda}$

$$\left(\frac{1}{\omega'} - \frac{1}{\omega}\right) = \frac{\hbar^{=\frac{h}{2\pi}}}{m_0c^2}(1-\cos\vartheta) = \frac{2\hbar}{m_0c^2}\sin^2\frac{\vartheta}{2}$$

$\frac{\lambda'}{2\pi c} - \frac{\lambda}{2\pi c}$

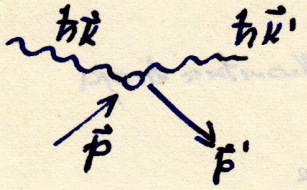
$$\Delta\lambda = \lambda' - \lambda = \frac{2h}{m_0c}\sin^2\frac{\vartheta}{2} \qquad \lambda_c \sim 2{,}4\cdot 10^{-10}\,\text{cm}$$

$$= 2\lambda_c\sin^2\frac{\vartheta}{2}$$

Compton-ova valna duljina

$E = c\sqrt{p^2 + m_0^2 c^2}$

Dz: (1) elektron se giba



$$\hbar\vec{k} + \vec{p} = \hbar\vec{k}' + \vec{p}'$$
$$\hbar\omega + E = \hbar\omega' + E'$$

provjeri za $E = 0$

Messiah I 5. spec: $p_e \parallel p = \hbar\vec{k} = \hbar\frac{\omega}{c}\hat{n}$

$E = \sqrt{m_0^2 c^4 + \hbar^2\omega^2}$

Dopler

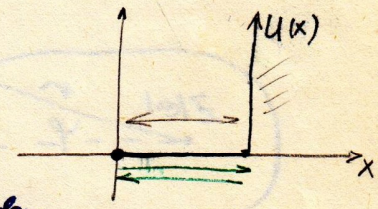
$$\Rightarrow \Delta\lambda = 2\lambda_c \frac{p_e + p}{E - p_e c}\sin^2\frac{\vartheta}{2} \Leftrightarrow \Delta\lambda = 2\lambda_c \frac{p_{el.}\cdot c}{E_{el.} - p_{el.} c}\sin^2\frac{\vartheta}{2}$$

Račun energije uz Sommerfeld-Wilson-ove uvjete kvantizacije

$$\oint p_i \, dq_i = n_i h$$

1) Iz kvantnih uvjeta izračunati energiju čestice kao se giba u potencijalnoj jami:

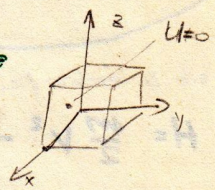
$$U(x,y,z) = \begin{cases} 0 & 0 \le x,y,z \le a \\ \infty & x,y,z \notin (0,a) \end{cases}$$



Konz. sile & nerelativ. fizika

jama sa ∞ visokim zidom

klasičan zid kroz koji čestice ne prolaze



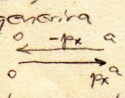
$$H = \frac{p^2}{2m} + U(\vec{r})$$

za $x,y,z \in (0,a)$ $\quad H = \frac{p^2}{2m} = \frac{1}{2m} S' p_x^2$

jama je kocka i isto za y, z

$$\oint p_x \, dx = n_x h \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = 0 \Rightarrow p_x = const$$

degenerira



a to je pozitivan smisao obilaženja napravca

$$\Rightarrow p_x \int_0^a dx + (-p_x)\int_a^0 dx = 2p_x a = n_x h$$

$$\Rightarrow \left.\begin{aligned} p_x &= \frac{n_x h}{2a} & n_x &= 0,1,2,\dots \\ p_y &= \frac{n_y h}{2a} & n_y &= 0,1,2,\dots \\ p_z &= \frac{n_z h}{2a} & n_z &= 0,1,2,\dots \end{aligned}\right\} \Rightarrow E = \frac{h^2}{8ma^2}(n_x^2 + n_y^2 + n_z^2)$$

n_x, n_y, n_z - neovisni

$E_{n_x n_y n_z}$ - energija čestice ovisi o tri kvantna broja

$$n^2 = n_x^2 + n_y^2 + n_z^2$$

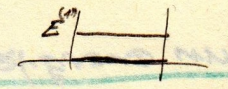
$$E_0 = 0 \qquad \Rightarrow E^{(n)} = \frac{h^2}{8ma^2} n^2$$

$$E^{(1)} = \frac{h^2}{8ma^2}$$

$n_x = 1$	$n_y = 0$	$n_z = 0$
$= 0$	1	0
$= 0$	0	1

energija čestice na ta tri načina

stanje te energije 3 puta degenerirano



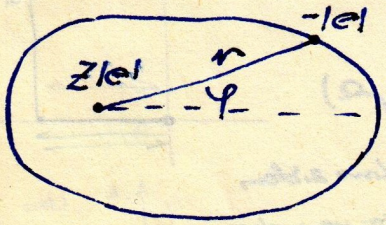
razon pojam iz kvantnoj stat. fizici

→ degenerirana stanja = ista energija, a različita valna fja.

detaljnije kod izv. Schröd. jbe.

2) Izračunati energiju elektrona koji se giba po eliptičkoj stazi oko jezgre naboja $Z|e|$ uz Sommerfeld-Wilson-ove uvjete!

→ gibanje u ravnini r, φ



i) $\oint p_r dr = n_r h$ - $n_r = 0, 1, 2, \ldots$

ii) $\oint p_\varphi d\varphi = n_\varphi h$ $\quad n_\varphi = 1, 2, 3, \ldots$

Kasnije opravdano

Konz. sile & nerelativ. fiz.

$$H = \frac{m}{2} v^2 - \frac{Z|e|^2}{r} \qquad \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\cdot\hat{r} + r\dot{\varphi}\hat{\varphi}$$

$L = T - U$

$$H = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{Ze^2}{r} \qquad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi}$$

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \quad \Rightarrow p_\varphi = \text{const.}$$

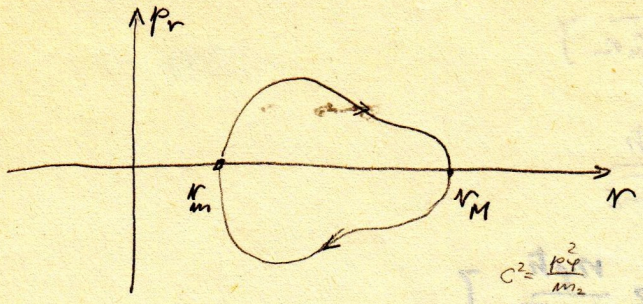
$$\Rightarrow n_\varphi h = \oint p_\varphi d\varphi = p_\varphi \cdot \int_0^{2\pi} d\varphi \quad \Rightarrow \quad p_\varphi = n_\varphi \cdot \hbar$$

$$E = H = \frac{1}{2m}\left(p_r^2 + \frac{n_\varphi^2 \hbar^2}{r^2}\right) - \frac{Ze^2}{r}$$

$$\Rightarrow p_r = \pm\left(2mE + \frac{2mZe^2}{r} - \frac{n_\varphi^2\hbar^2}{r^2}\right)^{1/2}$$

→ i) ovisit o energiju

stac. fja. s obzirom na ∞ r



$$p_r = \pm\left(2mE + \frac{2mZe^2}{r} - \frac{n_\varphi^2\hbar^2}{r^2}\right)^{1/2}$$

zahtjev $2mE + \frac{2mZe^2}{r} - \frac{n_\varphi^2\hbar^2}{r^2} \geq 0 \quad / \cdot r^2$

$$f(r) = 2mE\cdot r^2 + 2mZe^2\cdot r - n_\varphi^2\hbar^2 \geq 0$$

$$r_{1,2} = \frac{-2mZe^2 \pm \sqrt{4m^2Z^2e^4 + 8mEn_\varphi^2\hbar^2}}{4mE} < 2mZe^2 \quad \text{jer } E<0$$

$E<0 \Rightarrow r_{1,2} > 0$ što odgovara jer $r \geq 0$

$f(r) \geq 0$ za $r_{min} < r < r_M$

$$C^2 = \frac{p_\varphi^2}{m_2}$$

za elipsu $\varepsilon < 1 \Rightarrow E<0$

$$p = \frac{C^2}{|k|}$$

$$\varepsilon = \sqrt{1 + \frac{2EC^2}{\mu k^2}}$$

ili

$$r_{min} = \frac{p}{1+\varepsilon} \qquad r_{max} = \frac{p}{1-\varepsilon}$$

$$p_r = 0 \Rightarrow r_m, r_M$$

pozitivan smisao obilaženja

$$\oint p_r\,dr = n_r h \Rightarrow \int_{r_m}^{r_M} (\quad)^{1/2}dr + \int_{r_M}^{r_m} -(\quad)^{1/2}dr = 2\int_{r_m}^{r_M} (\quad)^{1/2}dr$$

jer $r\uparrow \Rightarrow p_r>0$ $\qquad$ $r\downarrow \Rightarrow p_r<0$

$$\Rightarrow 2\int_{r_m}^{r_M} (\quad)^{1/2}dr = n_r h$$

supstitucija: $t = \dfrac{2r - r_M - r_m}{r_M - r_m}$

$$p_r = \pm\left[\frac{1}{r^2}\cdot 2mE(r-r_m)(r-r_M)\right]^{1/2}$$

$$= \pm\frac{\sqrt{2mE}}{r}\cdot\frac{r_M - r_m}{2}\sqrt{t^2-1}$$

$$r = \frac{(r_M - r_m)}{2}t + \frac{(r_M + r_m)}{2}$$

$$r - r_m = \frac{r_M - r_m}{2}t + \frac{r_M - r_m}{2} = \frac{r_M - r_m}{2}(t+1)$$

$$r - r_M = \frac{r_M - r_m}{2}t + \frac{r_m - r_M}{2} = \frac{r_M - r_m}{2}(t-1)$$

$$\rightarrow dr = \frac{r_M - r_m}{2}dt$$

$$\frac{\cancel{r_M^2} + 2r_Mr_m + \cancel{r_m^2} - \cancel{r_M^2} + 2r_Mr_m - \cancel{r_m^2}}{(r_M - r_m)^2}$$

$$\int_{r_m}^{r_M} (\quad)^{1/2}dr = \int_{-1}^{1} \frac{\sqrt{2mE}}{\cancel{\frac{r_M - r_m}{2}}\left(t + \frac{r_M + r_m}{r_M - r_m}\right)}\cdot\cancel{\frac{r_M - r_m}{2}}\sqrt{t^2-1}\cdot\frac{r_M - r_m}{2}dt$$

$$t^2 = u^2 - 2\frac{r_M + r_m}{r_M - r_m}\cdot u + \left(\frac{r_M + r_m}{r_M - r_m}\right)^2$$

$$i\sqrt{-2mE} = \frac{\sqrt{2mE}\,(r_M - r_m)}{2}\int_{-1}^{1}\frac{\sqrt{t^2-1}}{t + \frac{r_M + r_m}{r_M - r_m}}dt$$

$$u = t + \frac{r_M + r_m}{r_M - r_m} \qquad t=-1 \quad u = \frac{2r_m}{r_M - r_m}$$

$$du = dt \qquad t=1 \quad u = \frac{2r_M}{r_M - r_m}$$

$$\int_{-1}^{1}\frac{\sqrt{t^2-1}}{t + \frac{r_M + r_m}{r_M - r_m}}dt = \int_{\frac{2r_m}{r_M - r_m}}^{\frac{2r_M}{r_M - r_m}} \frac{\sqrt{u^2 - 2\frac{r_M + r_m}{r_M - r_m}u + \frac{4r_Mr_m}{(r_M - r_m)^2}}}{u}du = \sqrt{X}$$

$a = 1 > 0$

$c > 0$

$$\Delta = 4ac - b^2 = \frac{16r_Mr_m - 4(r_M + r_m)^2}{(r_M - r_m)^2} = 4\frac{4r_Mr_m - r_M^2 - 2r_Mr_m - r_m^2}{(r_M - r_m)^2} = -4\frac{(r_M - r_m)^2}{(r_M - r_m)^2} = -4 < 0$$

$$= \cancel{\sqrt{t^2-1}\Big|_{-1}^{1}} - \frac{2}{2}\cdot\frac{r_M + r_m}{r_M - r_m}\int\frac{du}{\sqrt{\ }} + \frac{4r_Mr_m}{(r_M - r_m)^2}\int\frac{du}{u\sqrt{\ }} =$$

$$= -\frac{r_M + r_m}{r_M - r_m}\left[\ln\left(2\sqrt{X} + 2u - 2\frac{r_M + r_m}{r_M - r_m}\right)\right] + \frac{4r_Mr_m}{(r_M - r_m)^2}\left[-\frac{r_M - r_m}{2\sqrt{r_Mr_m}}\ln\left(\frac{r_M - r_m}{|r_M - r_m|}\frac{\sqrt{X}}{u} + \frac{8r_Mr_m}{(r_M - r_m)^2 u} - 2\frac{r_M + r_m}{r_M - r_m}\right)\right]$$

$$= -\frac{r_M + r_m}{r_M - r_m}\underbrace{\ln\frac{2}{-2}}_{-i\cdot\pi} - \frac{2\sqrt{r_Mr_m}}{(r_M - r_m)}\underbrace{\ln\frac{-2}{2}}_{-i\cdot\pi} =$$

$$= i\cdot\pi\,\frac{r_M + r_m + 2\sqrt{r_Mr_m}}{r_M - r_m}$$

$$\Rightarrow \int_{r_m}^{r_M} (\quad)^{1/2} dr = -\frac{\pi}{2}\sqrt{-2mE}\left[r_M + r_m + 2\sqrt{r_M r_m}\right]$$

$$r_M + r_m = -\frac{Ze^2}{E} \qquad \sqrt{r_M \cdot r_m} = \frac{n_\varphi \hbar}{\sqrt{-2mE}}$$

$$\Rightarrow n_r h = -\pi\sqrt{-2mE}\left[-\frac{Ze^2}{E} + 2\frac{n_\varphi \hbar}{\sqrt{-2mE}}\right]$$

$$2 n_r \hbar = +\frac{Ze^2}{E(\sqrt{-2mE})^{-1}} - 2\frac{n_\varphi \hbar}{1}$$

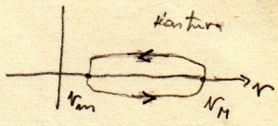
$$2(n_r + n_\varphi)\hbar = \frac{Ze^2\sqrt{-2mE}}{E} \quad /^2$$

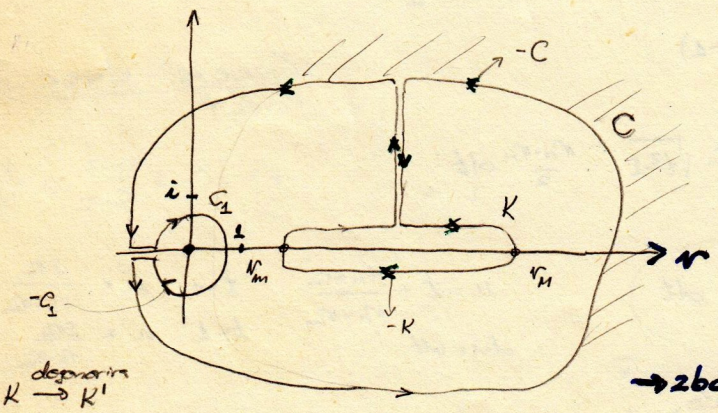
$$\cancel{4}(n_r + n_\varphi)^2 \hbar^2 = -\frac{\cancel{2}m Z^2 e^4 \cancel{E}}{E^{\cancel{2}}}$$

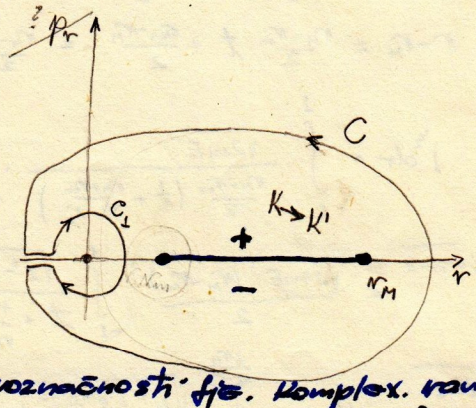
$$\Rightarrow E = -\frac{Z^2 e^4 m}{2\hbar^2 (n_r + n_\varphi)^2}$$

drugi način: $n_r h = \oint p_r dr \qquad p_r(r) = \pm\left(2mE + \frac{2mZe^2}{r} - \frac{n_\varphi^2 \hbar^2}{r^2}\right)^{1/2}$

-u kompleksnoj ravnini $\quad p_r(z) = \pm\left(2mE + \frac{2mZe^2}{z} - \frac{n_\varphi^2 \hbar^2}{z^2}\right)^{1/2}$







$K \xrightarrow{\text{degenerira}} K'$

$$\oint_{K'} = -\oint_{K} - \oint_{C} = \oint_{K+C} = 0$$

Cauchy-ov teorem

→ zbog dvoznačnosti fje. kompleks. ravnina se razreže u gornjoj kompleks. poluravnini (s Re osi) je $p_r(z) = +(\quad)^{1/2}$

u donjoj —"— je $p_r(z) = -(\quad)^{1/2}$

što odgovara činjenici da za $r \uparrow$ $p_r > 0$, a $r \downarrow$ $p_r < 0$

Kad K-ja K degenerira u dužinu $\overline{r_m r_M}$

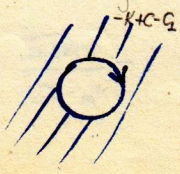
(*) $$\oint p_r dr = \int_{r_m}^{r_M} +(\quad)^{1/2} dr + \int_{r_M}^{r_m} -(\quad)^{1/2} dr = \oint_{-K'} p_r(z) dr = \oint_{-K} p_r(z) dz = \oint_{-C+C_1} p_r(z) dz = \oint_{-C} p_r(z) dz + \oint_{C_1} p_r(z) dz =$$

$-C$ - povoljnije nego

$$= 2\pi i \operatorname{Res}_{z=\infty} p_r(z) + 2\pi i \operatorname{Res}_{z=0} p_r(z)$$

to obuhvaća tj. promatramo res. iznutra

$$\oint_{-K+C-C_1} = 0 \Rightarrow \oint_{-K} + \oint_{C-C_1} = 0$$



promatramo res. izvan

$$\operatorname{Res}_{z=\infty} f(z) = \frac{1}{2\pi i}\oint_{-K} f(z) dz = -\frac{1}{2\pi i}\oint_{K} f(z) dz$$

K po velikoj polukružnici

$$n_r h = \oint_C p_r(z)\,dz \overset{(*)}{=} 2\pi i \left[\operatorname*{Res}_{z=0} p_r(z) + \operatorname*{Res}_{z=\infty} p_r(z) \right]$$

opć. $f(z)$ z_0 = pol.

$f(z) = \sum_{n=-\infty}^{\infty} A_n (z-z_0)^n$

$\operatorname{Res}_{z=z_0} f(z) = A_{-1}$

u gornjoj poluravni & Re osi

$\overset{?}{\to}$ naštimavanje - strože pokazati

$$p_r(z) = \pm \frac{\sqrt{-n_\varphi^2 \hbar^2}}{z} \left[1 - \frac{2mZe^2}{n_\varphi^2 \hbar^2} z - \frac{2mE}{n_\varphi^2 \hbar^2} z^2 \right]^{1/2}$$

u donjoj poluravni

dominantno u okolici nule

$$\Rightarrow \operatorname*{Res}_{z=0} p_r(z) = \pm \sqrt{-n_\varphi^2 \hbar^2}$$

jer 0 na realnoj osi

$$p_r(z) = \pm \sqrt{2mE} \left(1 + \frac{Ze^2}{E} \cdot \frac{1}{z} - \frac{n_\varphi^2 \hbar^2}{2mE} \cdot \frac{1}{z^2} \right)^{1/2}$$

$(1+x)^{1/2}$ dominantno za $z=\infty$ $\quad$ za $|z| \gg 1$

$$= \pm \sqrt{2mE} \left(1 + \frac{1}{2} \frac{Ze^2}{E} \cdot \frac{1}{z} - \frac{1}{2} \cdot \frac{1}{4} \left(\frac{Ze^2}{E} \right)^2 \cdot \frac{1}{z^2} + \dots \right)$$

$$\Rightarrow \operatorname*{Res}_{z=\infty} p_r(z) = \pm \sqrt{2mE} \cdot \frac{1}{2} \frac{Ze^2}{E}$$

moguće i jedan i drugi predznak pr $z=\infty$ se veže na da li je u donjoj ili gornjoj poluravni

$$\Rightarrow n_r h = 2\pi i \left[\sqrt{-n_\varphi^2 \hbar^2} \pm \frac{1}{2} \frac{Ze^2}{E} \sqrt{2mE} \right]$$

$$n_r \hbar = -n_\varphi \hbar \pm \frac{i}{2} Ze^2 \sqrt{\frac{2m}{E}}$$

$$\pm \frac{i}{2} \cdot \frac{Ze^2}{1} \cdot \sqrt{\frac{2m}{E}} = \hbar (n_r + n_\varphi) \quad /^2$$

$$-Z^2 e^4 \cdot \frac{2m}{4E} = \hbar^2 (n_r + n_\varphi)^2$$

$$E_{n_r n_\varphi} = -\frac{Z^2 e^4 m}{2\hbar^2 (n_r + n_\varphi)^2}$$

(*)

Mehanika matrica - princip korespodencije

$$[X, p] = i\hbar$$

$X = [(X_{mn})_{mn}]$

iz općenitog $[\hat{u}, \hat{v}] = \hat{u}\hat{v} - \hat{v}\hat{u} = i\hbar \{u, v\}_{q,p} = \sum_{j=1}^{s} \left(\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j} \right)$

kvantovalić su matrice

$\Rightarrow [\hat{q}_i, \hat{q}_j] = 0 \quad [\hat{p}_i, \hat{p}_j] = 0 \quad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$

$$X_{mn}(t) = a_{mn} e^{i\omega_{mn} t}$$

$$X_{mn} = X^*_{nm} \quad \text{pr} \quad a_{mn} = a^*_{nm} \quad \omega_{mn} = -\omega_{nm}$$

-konjug.

antisimetr.

1) Iz komutacionih pravila za koord. i impulse dokazati

$$\underbrace{\sum_k \frac{2m\omega_{kn}}{\hbar} |X_{mk}|^2}_{f_{mn}} = 1$$

f_{mn} – važno kod teorije diel. konst. – disperzije svjetlosti

ili $[p, x] = \frac{\hbar}{i} I$

$$[X, p] = i\hbar I$$

$[(X_{kn})_{kn}] \quad [(p_{mk})_{mk}]$

$X = [(X_{mk})_{mk}] \quad p = [(p_{kn})_{kn}]$

$\begin{pmatrix} x_{00} \\ x_{10} \\ x_{20} \end{pmatrix} \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \\ p_{20} & \end{pmatrix}$

$$X \cdot p - p \cdot X = i\hbar I$$

$$X \cdot p = [(\sum_{k=0} X_{mk} p_{kn})_{mn}]$$

$$p \cdot X = [(\sum_{k=0} p_{mk} X_{kn})_{mn}] \qquad I = [(\delta_{mn})_{mn}]$$

$$[X, p]_{mn} = \delta_{mn} \cdot i\hbar$$

$$\Rightarrow [X, p]_{mn} = \sum_{k=0}^{\infty} (X_{mk} p_{kn} - p_{mk} X_{kn}) = \delta_{mn}$$

$$(Xp)_{mn} - (pX)_{mn} = \delta_{mn} i\hbar$$

$$\Rightarrow \sum_{k=0}^{\infty} (X_{mk} p_{kn} - p_{mk} X_{kn}) = i\hbar\delta_{mn} = -\frac{\hbar}{i}\delta_{mn}$$

u klas. fizici kao i kvantnoj

$$p = m\dot{x} \Rightarrow p_{mn} = m\dot{x}_{mn} = i m \omega_{mn} X_{mn}$$

$a_{mn} e^{i\omega_{mn} t}$

$$\sum_{k=0}^{\infty} m i (\omega_{mk} X_{mk} X_{kn} - X_{mk} \omega_{kn} X_{kn}) = \frac{\hbar}{i}\delta_{mn}$$

masa

dijagonala $m = n \Rightarrow \delta_{mn} = 1$

$$\sum_{k=0} m (X_{mk} \omega_{kn} X_{kn} - \omega_{mk} X_{mk} X_{kn}) = \hbar$$

$X^*_{mk} \quad -\omega_{kn} \quad X^*_{mk}$

skalari

$$\sum_{k=0} m\omega_{kn} \cdot 2 |X_{mk}|^2 = \hbar \Rightarrow \sum_{k=0} \frac{2m\omega_{kn}}{\hbar} |X_{mk}|^2 = 1 \quad (*)$$

$|a_{mk}|^2$

$\ddot{x} = -\omega_0^2 x \Rightarrow x_n(t) = a_{-1}(n) e^{-i\omega_0 t} + a_1(n) e^{i\omega_0 t}$

stavimo $\omega(n) = \omega_0$

Primjer

2) Harmonički oscilator → nema gornjih titraja

$E_n = n\hbar\omega$ $\quad \omega_{nn'} = \frac{E_n - E_{n'}}{\hbar} = \underbrace{(n-n')}_{k}\omega$ $\quad$ princip korespondencije $\Rightarrow a_{nm} = 0$ za $m \neq n \pm 1$

$\Rightarrow a_{nm} \neq 0$ samo za $m = n \pm 1$

$x_{mn} = a_{mn} e^{i\omega_{mn} t}$

$$X = \begin{pmatrix} 0 & a_{01} e^{-i\omega_0 t} & 0 & \cdots \\ a_{10} e^{i\omega_0 t} & 0 & a_{12} e^{-i\omega_0 t} & \cdots \\ 0 & a_{21} e^{i\omega_0 t} & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

$\omega_{10} = 1 \cdot \omega(1)$

$a_{n-1\,n} e^{-i\omega_0 t}$

$a_{n+1\,n} e^{i\omega_0 t}$

$$\sum_{k=0} \frac{2m\omega_{kn}}{\hbar} |x_{nk}|^2 = 1$$

princip korespo:
$\omega_{nn-k} = k\omega(n)$
$a_{nn-k} = a_k(n)$

k & n se razlikuju samo za $\pm 1 = k$

$$\Rightarrow \frac{2m\omega_{n-1\,n}}{\hbar} |a_{n\,n-1}|^2 + \frac{2m\omega_{n+1\,n}}{\hbar} |a_{n\,n+1}|^2 = 1$$

$$-\frac{2m\omega_{n\,n-1}}{\hbar} |a_{n\,n-1}|^2 - \frac{2m\omega_{n\,n+1}}{\hbar} |a_{n\,n+1}|^2 = 1$$

$\omega_{nn+1} = -\omega_0 = (-1)\omega(n)$

$\omega_{nn-1} = \omega_0 = 1 \cdot \omega(n)$

? 1° Newton-ov zakon ne dostaje već je potrebno još dati komutacioni zakon za p & x

$\ddot{x} = -\omega_0^2 x \Rightarrow a_{nm}(\omega_{nm}^2 - \omega_0^2) = 0 \Rightarrow \omega_{nm} = \pm\omega_0$
$\omega_{nm} = 0 \Rightarrow a_{nm} = 0$

$$\frac{2m}{\hbar}\left[-\omega_0 |a_{n\,n-1}|^2 + \omega_0 |a_{n\,n+1}|^2\right] = 1$$

$$\Rightarrow |a_{n\,n+1}|^2 - |a_{n\,n-1}|^2 = \frac{\hbar}{2m\omega_0} \quad (**)$$

rekurzivna formula

$|a_{n\,n+1}|^2 = \frac{\hbar}{2m\omega_0} + |a_{n\,n-1}|^2 = \frac{\hbar}{2m\omega_0} + |a_{n-1\,n}|^2$

$a_{nm}^* = a_{mn}$
$a_{mn} = a_{nm}^*$

$a_{0-1} = a_1(0)$

→ pretpostavimo $E_0 = 0 \cdot \hbar\omega$

! $a_{0-1} = 0 \Rightarrow |a_{10}|^2 = |a_{01}|^2 = \frac{\hbar}{2m\omega_0}$

tj. ima smisla da osil. gibanje

$|a_{12}|^2 = \frac{\hbar}{2m\omega_0} + |a_{10}|^2 = \frac{\hbar}{2m\omega_0} + |a_{01}|^2$

$|a_{mn}|^2 = |a_{nm}|^2$

$|a_{12}|^2 = \frac{\hbar}{2m\omega_0} \cdot 2$

$|a_{n-1\,n}|^2 = \frac{\hbar}{2m\omega_0} \cdot n$

$\vdots$

$$|a_{n+1\,n}|^2 = |a_{n\,n+1}|^2 = \frac{\hbar}{2m\omega_0} + |a_{n-1\,(n-1)+1}|^2 = \frac{\hbar}{2m\omega_0} \cdot (n+1)$$

$$\Rightarrow a_{n\,n+1} = \sqrt{\frac{\hbar}{2m\omega_0}} \cdot \sqrt{n+1} \quad (**)$$

može biti i opći faktor: tj. kompleksan

Obrat: $x_{mn} = a_{mn} e^{i\omega_{mn} t}$

sa $a_{n\,n+1} = \sqrt{\frac{\hbar}{2m\omega_0}} \sqrt{n+1}$ $\qquad \omega_{n\,n+1} = -\omega_0$

za realne $a_{n\,n-1} = a_{n-1\,n} = \sqrt{\frac{\hbar}{2m\omega_0}} \sqrt{n}$ $\qquad \omega_{n\,n-1} = \omega_0$

tr. $a_{nm} = a_{nm}^*$

$a_{nm} = 0 \quad m \neq n \pm 1$

zadovoljava (*) $\quad a_{n\,n-1}^2 = a_{n+1\,n}^2$

$$-\sum_k \frac{2m\omega_{nk}}{\hbar} |a_{nk}|^2 = 1 \qquad -\frac{2m}{\hbar}\left[\omega_0 \frac{\hbar}{2m\omega_0} \cdot n - \omega_0 \frac{\hbar}{2m\omega_0} \cdot (n+1)\right] = (n+1) - n = 1$$

Ia. 1) Za H.O. izračunati matrice ~~za~~ kin. energije T i pot. energije U

Kin. energija:

$$T_{mn} = \left(\frac{p^2}{2m}\right)_{mn} \qquad p_{mn} = m\dot{x}_{mn} = m\cdot i\omega_{mn} a_{mn} e^{i\omega_{mn}t} = im\omega_{mn} x_{mn}$$

$$T_{mn} = \sum_l \frac{1}{2m} p_{ml}\cdot p_{ln} = \begin{cases} 0 & m\neq n \\ T_{nn} & m=n \end{cases}$$

!!! pravdi: općenito se pokazuje da je H dijagon. matrica $\Leftrightarrow$ konst. gibanja

Sapek II -21
Fermi 155-159

princip korresp. & harm. oscil.

$\Rightarrow l=n-1 \quad l=n+1$

$\omega_{nn-1} = \omega_0 \quad \omega_{nn+1} = -\omega_0$

$\omega_{mn} = -\omega_{nm}$

(2) $x_{ml}\cdot x_{ln} = a_{ml}\cdot a_{ln} e^{i(\omega_{ml}+\omega_{ln})t} = |a_{ln}|^2\delta_{mn}$

ili $a_{ml}\, a_{ln} = |a_{ln}|^2\delta_{mn}$

$$T_{nn} = \frac{1}{2m}\Big\{\Big[\big(mi\,\omega_{nn-1}a_{nn-1}e^{i\omega_{nn-1}t}\big)\big(mi(-\omega_{nn-1})\overset{a^*_{nn-1}}{a_{n-1n}}e^{i\overset{=\omega_{nn-1}}{\omega_{n-1n}}t}\big) + \big(mi\,\omega_{nn+1}a_{nn+1}e^{i\omega_{nn+1}t}\big)\big(mi\,\overset{-\omega_{nn+1}}{\omega_{n+1n}}a_{n+1n}e^{i\omega_{n+1n}t}\big)\Big]\Big\}$$

(**)

$$T_{nn} = \frac{m\omega_0^2}{2}\Big\{\overset{|a_{n-1n}|^2}{|a_{nn-1}|^2} + |a_{nn+1}|^2\Big\} \overset{(***)}{=} \frac{\omega_0\hbar}{4}(2n+1)$$

pot. energija:

$$U_{mn} = \left(\frac{m\omega_0^2}{2}x^2\right)_{mn}$$

$$U_{mn} = \sum_l \frac{m\omega_0^2}{2} x_{ml}x_{ln} = \begin{cases} 0 & m\neq n \\ U_{nn} & m=n \end{cases}$$

!!!

(2) $x_{ml}\cdot x_{ln} = |a_{ln}|^2\delta_{mn}$

(**)

$$U_{nn} = \frac{m\omega_0^2}{2}\Big\{\big(a_{nn-1}e^{i\omega_{nn-1}t}\big)\big(\overset{a^*_{nn-1}}{a_{n-1n}}e^{i\overset{-\omega_{nn-1}}{\omega_{n-1n}}t}\big) + \big(a_{nn+1}e^{i\omega_{nn+1}t}\big)\big(a_{n+1n}e^{i\omega_{n+1n}t}\big)\Big\}$$

$$= \frac{m\omega_0^2}{2}\underbrace{\Big\{|a_{nn-1}|^2 + |a_{nn+1}|^2\Big\}}_{\frac{\hbar}{2m\omega_0}(2n+1)} = \frac{\omega_0\hbar}{4}(2n+1)$$

$$H_{mn} = \underbrace{\left(\frac{p^2}{2m} + \frac{m\omega_0^2x^2}{2}\right)_{mn}}_{H\ \text{matrica}} \overset{\text{linearnost}}{\underset{\text{zbrajanje matrica}}{=}} T_{nn} + U_{nn} = m\omega_0^2\Big\{|a_{nn-1}|^2 + |a_{nn+1}|^2\Big\}$$

$$T = U = \frac{\hbar\omega_0}{4}\begin{pmatrix} 1 & 0 & \cdots \\ 0 & 3 & \\ \vdots & & 5 \\ 0 & & & \ddots \end{pmatrix} \qquad H = \frac{\hbar\omega_0}{2}\begin{bmatrix} 1 & & \cdots 0 \\ & 3 & \\ 0 & & 5 \\ & & & \ddots \end{bmatrix}$$

Teorija angularnog momenta

$$\vec{M} = \vec{r} \times \vec{p}$$

Komutaciona pravila

$$[M_x, M_y] = i\hbar M_z \quad \text{cikliški}$$

$[\hbar L_x, \hbar L_y] = i\hbar\, \hbar L_z$

$\hbar^2 (L_x L_y - L_y L_x)$

$$\vec{M} \overset{def}{=} \hbar \vec{L} \quad \text{i mjerimo u jedinicama } \hbar$$

cm g cm/s = g cm²/s²

erg s

$$[L_x, L_y] = i L_z \quad \text{cikliški}$$

Kako izgleda matrica mom. impulsa ?

(1) $[L^2, L_i] = 0$

$$[L_x^2 + L_y^2 + L_z^2, L_x] = [L_y^2 + L_z^2, L_x] =$$

$$[L_y^2, L_x] = L_y \underbrace{[L_y, L_x]}_{-iL_z} + \underbrace{[L_y, L_x]}_{-iL_z} L_y \quad \text{i}$$

$$[L_z^2, L_x] = L_z \underbrace{[L_z, L_x]}_{iL_y} + \underbrace{[L_z, L_x]}_{iL_y} L_z$$

$$= -iL_y L_z - iL_z L_y + iL_z L_y + iL_y L_z = 0$$

nul matrica

Def: $L_+ = L_x + iL_y$, $L_- = L_x - iL_y$ $\Rightarrow L_- = L_+^\dagger$

$L_+ = [(\mu_{mn})_{mn}]$ $\quad L_- = [(\bar{\mu}_{mn})_{mn}]$

herm. matrica

$\underline{\mu_{mn}^*} = (\mu_x)^*_{mn} + i^* (\mu_y)^*_{mn} = \mu_{x\,mn} - i\mu_{y\,mn} = \underline{\bar{\mu}_{mn}}$

(2) $[L^2, L_\pm] = 0$ → ako dva operatora komutiraju znači da imaju zaj. svojstvene fje. (~~vrijednosti~~) (vektore)

dokazano 7. org.

?

$$[L_z, L_+] = iL_y + i(-iL_x) = L_+$$

$$[L_z, L_-] = -L_-$$

$$[L_+, L_-] = [L_x, L_x] - i[L_x, L_y] - i[L_x, L_y] + [L_y, L_y]$$

$$= -2i[L_x, L_y] = 2L_z$$

ne: komutiraju ili ne ne ne ne impulsi

$$\Rightarrow \quad L_z = \frac{1}{2}\left(L_+ \cdot L_- - L_- \cdot L_+\right)$$

$L_+ = L_x + iL_y \qquad L_+ + L_- = 2L_x$

$L_- = L_x - iL_y \qquad L_+ - L_- = 2iL_y$

$$L^2 = L_x^2 + L_y^2 + L_z^2 = \frac{1}{4}\left[\cancel{L_+^2} + L_- \cdot L_+ + L_+ \cdot L_- + \cancel{L_-^2} - \cancel{L_+^2} + L_- \cdot L_+ + L_+ \cdot L_- - \cancel{L_-^2}\right] + L_z^2$$

$$L^2 = \frac{1}{2}\left[L_+ \cdot L_- + L_- \cdot L_+\right] + L_z^2$$

$$\left.\begin{aligned} L^2 + L_z &= L_+ \cdot L_- + L_z^2 \\ L^2 - L_z &= L_- \cdot L_+ + L_z^2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} L_+ \cdot L_- &= L^2 - L_z^2 + L_z \\ L_- \cdot L_+ &= L^2 - L_z^2 - L_z \end{aligned}$$

Komp. angul. mom. ne komutiraju sve jedna s drugom, pa sve nisu dijag. matrice

→ možemo odabrati:

$L_z = (\backslash)$ dijagonalna

a L^2 sa svima komutira možemo je odabrati dijagonalnom

$[L^2, L_z] = 0 \qquad L^2 = (\backslash)$

a komutiraju inače, a ne samo što su odabrane dijagonalnim

L_z & L^2 međusobom komutiraju ! (*) Koji ⇒ možemo uzeti da imaju iste vlastite svojstvene vektore

$|lm\rangle$ Schrödinger → $Y_l^m(\vartheta, \varphi)$

→ tj. pretpostavka da smo uzeli takvu bazu u (Hilbertovom) prostoru fja da

L^2 djeluje na fju. $|lm\rangle$ = broj. $\cdot \underbrace{|lm\rangle}_{\text{fja.}}$

→ Kakve su svojstvene vrijednosti (od svojst. vektora - fja) tj. dijagonalni elementi tih operatora

dokaz (*)

$[L^2, L_z] = 0$ & pretp. $|lm\rangle$ svojstvena za L^2

$L^2 L_z |lm\rangle = L_z L^2 |lm\rangle$

$L^2 \underbrace{L_z |lm\rangle}_{\text{svojstvena za } L^2} = l(l+1) \underbrace{L_z |lm\rangle}$

pa moramo dobiti i sam $|lm\rangle \Rightarrow |lm\rangle$ svojstv. i za L_z

→ pretpostavka:

$$L^2 |lm\rangle = l(l+1) \overset{\text{odgovara}}{E} |lm\rangle$$

→ stavili $l(l+1)$ koje odgovara l-u u $|lm\rangle$ iz raznih razloga

$$L_z |lm\rangle = m \overset{?}{E} |lm\rangle \quad \text{gdje} \quad E = [(\delta_{nm})_{nm}]$$

bitno: ne postavljamo da su m i l cijeli brojevi: već mogu biti bilo kakvi ili tek kasnije pokazujemo kakvi jedino mogu biti.

$$L_z^2 |lm\rangle = L_z \{L_z |lm\rangle\} = L_z \{m |lm\rangle\} = m \cdot (m |lm\rangle) = m^2 E |lm\rangle$$

$L^2 = l(l+1) \cdot E \qquad L_z = \begin{pmatrix} -l & & 0 \\ & -l+1 & \\ & & \ddots \\ 0 & & l \end{pmatrix} \qquad |lm\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} -l \\ \\ \\ \leftarrow m \\ \\ l \end{matrix} \qquad m = -l, \dots 0, \dots l$

iz pretp. slijedi:

$$L_- L_+ |lm\rangle = (L^2 - L_z^2 - L_z)|lm\rangle$$

$$= [l(l+1) - m^2 - m]E|lm\rangle$$

$(l-m)(l+m+1)$

za $m=l \Rightarrow L_- L_+ |lm\rangle = 0 = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$

$\Rightarrow L_+ |lm\rangle = 0 = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$

$$L_+ L_- |lm\rangle = (L^2 - L_z^2 + L_z)|lm\rangle$$

$$= [l(l+1) - m^2 + m]|lm\rangle$$

$$= (l+m)(l-m+1)E|lm\rangle$$

jer ako $L_- \neq 0$ matrica dl. mora biti $\det L_- = 0$

$$\begin{pmatrix}0 & \cdots & 0\\ \vdots & & \vdots\\ 0 & \cdots & 0\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_{2l+1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \xRightarrow{\det L \neq 0} \begin{pmatrix}x_1\\ \vdots\\ x_{2l+1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

za $m=-l \Rightarrow L_+ L_- |lm\rangle = 0$

$\Rightarrow L_- |lm\rangle = 0$

nemora

(*)

→ matrični elementi operatora $L_- L_+$

$$0 \leq \underbrace{\langle lm| \cdot L_- L_+ |lm\rangle}_{\|L_+|lm\rangle\|^2} = \langle lm| \cdot (l-m)(l+m+1)|lm\rangle = (l-m)(l+m+1)\langle lm|lm\rangle$$

$(0 \ldots 1 \ldots 0)\, L_-$

kao skalarni produkt vektora sa samim sobom

$$= (l-m)(l+m+1) \Rightarrow m \leq l$$

pokazano na 8.

$l-m \geq 0 \Rightarrow m \leq l$

$(l+m+1) \geq 0 \Rightarrow m \geq -l-1$

$l-m \leq 0 \Rightarrow m \geq l$

$(l+m+1) \leq 0 \Rightarrow m \leq -l-1$

→ skalarni produkt u prostoru Kuglinih fja.

tipa $\int d\Omega\, (L_+ Y_l^m)(L_+ Y_l^m)^* = \int d\Omega\, Y_l^{m*} L_- L_+ Y_l^m$!!! napro

→ matrični elem. oper. $L_+ L_-$

$$0 \leq \underbrace{\langle lm| L_+ L_- |lm\rangle}_{\|L_-|lm\rangle\|^2} = (l+m)(l-m+1) \Rightarrow m \geq -l$$

$\|L_-|lm\rangle\|^2 = [L_-|lm\rangle]^\dagger [L_-|lm\rangle] = |lm\rangle^\dagger L_-^\dagger L_- |lm\rangle = \langle lm| L_+ L_- |lm\rangle$

matrica

$l+m \geq 0 \Rightarrow m \geq -l$

$l-m+1 \geq 0 \Rightarrow m \leq l+1$

točnije:

(*) (11.0) ⇒ $L_+ |lm\rangle = 0$ za $m=l$ jer tada $\|L_+|lm\rangle\| = 0$

ili (*) $L_- |lm\rangle = 0$ za $m=-l$ $\|L_-|lm\rangle\| = 0$

→ Što slijedi iz? $[L^2, L_\pm] = 0$ napadamo $|lm\rangle$

(**) napat

$$L^2 L_+ |lm\rangle - L_+ \underbrace{L^2 |lm\rangle}_{l(l+1)E|lm\rangle} = 0$$

$$L^2 L_+ |lm\rangle = L_+ l(l+1)|lm\rangle$$

$$L^2 \underbrace{L_+ |lm\rangle} = l(l+1) \underbrace{L_+ |lm\rangle}$$

opet svojstven vektor operatora L^2 kao i $|lm\rangle$

original!!! pa $L_+ |lm\rangle$ mora dati i sam $|lm\rangle$ tj. tada zaključujemo da je $|lm\rangle$ svojstven vektor i za L_+

$\Rightarrow$ djelovanjem L_+ na $|lm\rangle$ možemo dobiti sve ostale vlastite vektore od L^2

(*) $[L_z, L_+] = L_+ \quad / |lm\rangle$

$$L_z \cdot L_+ |lm\rangle - \overbrace{L_+ \cdot L_z |lm\rangle}^{m|lm\rangle} = L_+ |lm\rangle$$

(*)

$$\underbrace{L_z L_+ |lm\rangle}_{\text{vlastiti vektor za } L_z \text{ s vl. vrij. } (m+1)} = (m+1) L_+ |lm\rangle \quad \text{i)}$$

$\Rightarrow$ usporedimo ℓ_i sa $L_z |lm\rangle = m|lm\rangle$

$\Rightarrow L_+ |lm\rangle \sim |l\, m+1\rangle$.

(**) $[L_z, L_-] = -L_- \quad / |lm\rangle$

$$L_z L_- |lm\rangle - \overbrace{L_- L_z |lm\rangle}^{m|lm\rangle} = -L_- |lm\rangle$$

$$\underbrace{L_z L_- |lm\rangle}_{\text{vl. vektor za } L_z \text{ s vl. mj. } (m-1)} = (m-1) L_- |lm\rangle \quad \text{ii)}$$

$\Rightarrow L_- |lm\rangle \sim |l\, m-1\rangle$

postupak: odabarano jedan vl. vektor $|lm\rangle$ i djelujemo L_+

origin.!!! matem. indukcijom: $[L_z, L_+^p] = p L_+^p$ (*) $[L_z, L_+^{p+1}] = L_z L_+^{p+1} - L_+^{p+1} L_z = \underbrace{L_z L_+}\cdot L_+^p - L_+^{p+1} L_z \overset{(*)}{=}$

$= (L_+ + L_+ L_z) L_+^p - L_+^{p+1} L_z = L_+^{p+1} + L_+ (L_z L_+^p - L_+^p L_z)$

$\overset{(*)}{=} L_+^{p+1} + p L_+^{p+1} = (p+1) L_+^{p+1}$

Q.E.D.

$$L_z L_+^2 |lm\rangle = 2 L_+^2 |lm\rangle + L_+^2 \underbrace{L_z |lm\rangle}_{m|lm\rangle}$$

$$= (m+2) L_+^2 |lm\rangle$$

$\vdots$

$$L_z L_+^p |lm\rangle \overset{(*)}{=} p L_+^p |lm\rangle + L_+^p \overbrace{L_z |lm\rangle}^{m|lm\rangle}$$

$$= (m+p) L_+^p |lm\rangle \overset{?}{=} 0$$

gdje: $m = -\ell, \dots 0, \dots \ell$

$p = 0, 1, 2, \dots (\ell - m)$ & $\ell = \cancel{0, 1, 2, 3, \dots}$

$L_+^0 = E \quad \Rightarrow L_z |lm\rangle = m \cdot |lm\rangle$

$\Rightarrow L_+^p |lm\rangle \sim |l\, m+p\rangle$

zaustavi p, jer je $L_+ |lm\rangle = 0$ $m = \ell$

$L_+^{\ell - m} |lm\rangle \sim |l\, \ell\rangle$ podsjeća na $S_\ell^{(m)}(\xi) = \frac{(-)^{\ell+m} (\ell+m)!}{2^\ell \cdot \ell! \, (\ell-m)!} \frac{d^{\ell-m}}{d\xi^{\ell-m}} (1-\xi^2)^\ell$

paper (***)

$\Rightarrow m + p_{max} = \ell \quad \Rightarrow 0 \le p_{max} = \ell - m \le 2\ell$ (iii) za $-\ell \le m \le \ell$

$\Rightarrow p = 0, 1, 2, \dots (\ell - m)$

!!! od ongrno

matem. indukcijom: $[L_z, L_-^q] = -q \cdot L_-^q$ (**) $[L_z, L_-^{q+1}] = L_z L_-^{q+1} - L_-^{q+1} \cdot L_z = L_z \cdot L_- \cdot L_-^q - L_-^{q+1} \cdot L_z \overset{(**)}{=}$

$= (-L_- + L_- L_z) \cdot L_-^q - L_-^{q+1} \cdot L_z = -L_-^{q+1} + L_- (L_z L_-^q - L_-^q L_z) =$

$\overset{(**)}{=} -L_-^{q+1} - q \cdot L_-^{q+1} = -(q+1) L_-^{q+1}$

$$\Rightarrow \underline{L_z L_-^q |lm\rangle} = -q L_-^q |lm\rangle + L_-^q \cdot \underbrace{L_z |lm\rangle}_{m|lm\rangle}$$

$$= \underline{(m-q) L_-^q |lm\rangle}$$

$q = 0, 1, 2, 3, \ldots$

gdje $L_-^0 = E \Rightarrow L_z |lm\rangle = m|lm\rangle$

$$\Rightarrow L_-^q |lm\rangle \sim |l\ m-q\rangle$$

zahtjev:

$|l\ m-q\rangle \neq 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow -l \leq m-q \leq l \Rightarrow m+l \geq q \geq m-l$

& $q \geq 0$

& $-l \leq m \leq l \Rightarrow -2l \leq m-l \leq 0$

$0 \leq m+l \leq 2l$

$\Rightarrow 2l \geq m+l \geq q \geq 0 \geq m-l \geq -2l$

$\Rightarrow q_{max} = m+l$

$$\Rightarrow \underline{0 \leq q_{max} = l+m \leq 2l} \quad (i)$$

$\Rightarrow q = 0, 1, 2, \ldots (l+m)$

$\underline{q, p \in N \cup \{0\}}$

$q = l+m$

$p = l-m$

$\Rightarrow p_{max} + q_{max} = 2l$

$\underline{q_{max} - p_{max} = 2m}$

$p=0\ q=0$

$q=0\ p=1$
$q=1\ p=0$

$q=0\ p=2$
$q=1\ p=1$
$q=2\ p=0$

$q=0\ p=3$
$q=1\ p=2$
$q=2\ p=1$
$q=3\ p=0$

$q=0\ p=4$
$q=1\ p=3$
$q=2\ p=2$
$q=3\ p=1$
$q=4\ p=0$

$$\Rightarrow \underline{l} = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$$

$$\underline{m} = 0, \begin{matrix} -\frac{1}{2} \\ \frac{1}{2} \end{matrix}, \begin{matrix} -1 \\ 0 \\ 1 \end{matrix}, \begin{matrix} -\frac{3}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ +\frac{3}{2} \end{matrix}, \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix}$$

Supek II 30.

→ dakle, komutacioni pravila zadovoljena za kompon. mom. impulsa i za poluciele vrijednosti l

→ iz komutacionih pravila zaključujemo da je angular. moment kvantiziran

$L_z L_+ |lm\rangle = (m+1) L_+ |lm\rangle$

$(AB)^+ = B^+ A^+$

$$\rightarrow L_+ |lm\rangle = \underset{\text{skal. matrica}}{a} |l\ m+1\rangle$$

Koristimo $L_- = L^+$

$L_+ |lm\rangle = (\quad)(\quad) = (\quad)$

zahtijevamo: budući da je taj vektor normiran (tj. ima normu) odredimo $\underline{\underline{a}}$

$|lm\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} -l \\ \\ \leftarrow m \\ \\ l \end{matrix} \Rightarrow |lm\rangle^+ = \langle lm| = (0 \cdots 1 \cdots 0)$ (1 na m-tom mjestu)

$\| L_+ |lm\rangle \|^2 = (\quad)(\quad) = [L_+ |lm\rangle]^+ [L_+ |lm\rangle] = |lm\rangle^+ L_+^+ \cdot L_+ |lm\rangle = \langle lm| L_- L_+ |lm\rangle$

& $\| L_+ |lm\rangle \|^2 = [a |lm+1\rangle]^+ [a |lm+1\rangle] = \langle l\ m+1| a^+ a |l\ m+1\rangle = a^2 (1)$

$$\| L_+ |lm\rangle \|^2 = \underbrace{\langle lm| L_- L_+ |lm\rangle}_{(l-m)(l+m+1)} = a^2 \underbrace{\langle l\ m+1 | l\ m+1\rangle}_{1}$$

$$\Rightarrow a = \sqrt{(l-m)(l+m+1)}$$

$$\begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \leftarrow m+1$$

$$\Rightarrow \quad L_+ |l\, m\rangle = \sqrt{(l-m)(l+m+1)}\; |l\, m+1\rangle = \left[\left(\sqrt{(l-m)(l+m+1)} \cdot \delta_{n\, m+1} \right)_{n1} \right]$$

$$L_- |l\, m\rangle = \sqrt{(l+m)(l-m+1)}\; |l\, m-1\rangle = \left[\left(\sqrt{(l+m)(l-m+1)}\; \delta_{n\, m-1} \right)_{n1} \right]$$

→ matr. elementi operatora L_+ & L_-

$$L_+ |l m\rangle = \underbrace{\left[(\mu_{nk})_{nk} \right]}_{\text{tipa } (2l+1,\, 2l+1)} \underbrace{\left[(\delta_{mk})_{k1} \right]}_{(2l+1,\, 1)} = \left[\left(\sum_k \mu_{nk} \delta_{mk} \right)_{n1} \right] = \left[(\mu_{nm})_{n1} \right]$$

$$\Rightarrow \quad \mu_{nm} = \sqrt{(l-m)(l+m+1)}\; \delta_{n\, m+1} \quad \Rightarrow (L_+)_{m+1\, m} = \mu_{m+1\, m} = \sqrt{(l-m)(l+m+1)}$$

$$\mu_{nm} = 0 \qquad n \neq m+1$$

$$\Rightarrow \quad \mu_{nm} = \sqrt{(l+m)(l-m+1)}\; \delta_{n\, m-1} \quad \Rightarrow (L_-)_{m-1\, m} = \mu_{m-1\, m} = \sqrt{(l+m)(l-m+1)}$$

$$\mu_{nm} = 0 \qquad n \neq m-1$$

$$L_z |l m\rangle = m |l m\rangle \qquad \left[\left((L_z)_{nm} \right)_{n1} \right] = \left[(m \delta_{nm})_{n1} \right]$$

$$\& \quad (L_z)_{nm} = m \quad \text{dijagon.}$$

→ te tri relacije odreduju sve ostalo ! matrice angbl. mom. itd.

$$\vec{M} = \vec{r} \times \vec{p} \qquad \vec{M} = \hbar \vec{L} \qquad [L_i, L_j] = i \epsilon_{ijk} L_k$$

ciklički valjda označava

→ zadani l

1) $-l \leq m \leq l$ $\quad 2l+1$ vrij. $\quad L_z |l m\rangle = m |l m\rangle$

2) $(2l+1)$ lin. nezav. fja. $|l m\rangle$ – vektora

razapinju $(2l+1)$ dim. prostor

$$\rightarrow (L_i)_{nm} = \begin{array}{c} n=l \quad m=l \qquad\qquad m=-l \\ \begin{pmatrix} \\ \\ \\ \\ \\ \end{pmatrix} \\ n=-l \end{array}$$

takav dogovor

1. stupac $m=l$ max.

→ Definirali smo: $L_\pm = L_x \pm i L_y$

i pokazali:

$$L_+ |l m\rangle = \sqrt{(l-m)(l+m+1)}\ |l\ m+1\rangle$$
$$L_- |l m\rangle = \sqrt{(l+m)(l-m+1)}\ |l\ m-1\rangle$$

nisu dijag. matrice

→ vidjeti da se da su vektori baze $(2l+1)$ dim. Hilb prostora

za $m \in \mathbb{Z}$ $\quad m \to \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_l = |l m\rangle = Y_l^m(\vartheta,\varphi) = e^{im\varphi} \cdot P_l^{(m)}(\cos\vartheta) = e^{im\varphi} \sin^{-m}\vartheta\ \mathcal{P}_l^{(m)}(\cos\vartheta)$

faktor normiranja $\qquad l = 0, 1, 2, \ldots \qquad m = -l, \ldots 0, \ldots l$

$$\| |l m\rangle \|^2 = [|l m\rangle]^\dagger [|l m\rangle] = (0 \ldots \overset{m}{1} \ldots 0) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = (1)_{11} =$$

$2\pi \sin\vartheta\, d\vartheta$

$$= \langle l m | l m\rangle = \int d\Omega\ Y_l^m \cdot Y_l^{m*} = \int d\Omega\ P_l^{(m)}(\cos\vartheta) \cdot P_l^m(\cos\vartheta)$$

$\xi = \cos\vartheta$
$d\xi = -\sin\vartheta \cdot d\vartheta$

$$= 2\pi \int_0^\pi \sin\vartheta \left[P_l^{(m)}(\cos\vartheta)\right]^2 d\vartheta$$
$$= -2\pi \int_1^{-1} \left[P_l^m(\xi)\right]^2 d\xi = 2\pi \cdot \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!}$$

$$F_{m'm} = \langle l m' | F | l m\rangle \overset{\text{Dirac}}{=} \int d\Omega\ Y_l^{m'*} F Y_l^m$$

operator: $F \cdot |l m\rangle = [(f_{nk})_{nk}] \cdot [(\delta_{mk})_{k1}] = [(\sum_k f_{nk} \delta_{mk})_{n1}] = [(f_{nm})_{n1}] = \begin{pmatrix} f_{1m} \\ \vdots \end{pmatrix} \leftarrow n$

tipa $(2l+1, 2l+1)$ $\quad (2l+1, 1)$ $\quad$ tipa $(2l+1, 1)$

→ za dani l $\quad$ matrični elementi $\quad (\overset{n}{\downarrow}\ \overset{m}{\downarrow})$

$$\Rightarrow \langle l m' | F | l m\rangle = [(\delta_{nm'})_{1n}] [(f_{nm})_{n1}] = [(\sum_n f_{nm} \delta_{nm'})_{11}] = f_{m'm}$$

$$(L_z)_{nm} = \langle l n | \underbrace{L_z | l m\rangle}_{m|lm\rangle} = m \langle l n | l m\rangle = m \cdot \delta_{nm}$$

odgovara L_z

gdje: $n = l, \ldots -l$
$m = l, \ldots -l$
$\Rightarrow L_z = \begin{pmatrix} l & & & \\ & l-1 & & \\ & & \ddots & \\ & & & -l \end{pmatrix}$

$$(L^2)_{nm} = \langle l n | \underbrace{L^2 | l m\rangle}_{l(l+1)|lm\rangle} = l(l+1) \langle l n | l m\rangle = l(l+1)\, \delta_{nm}$$

$$(L_+)_{nm} = \langle l n | L_+ | l m\rangle = \sqrt{(l-m)(l+m+1)}\ \langle l n | l\ m+1\rangle = \sqrt{(l-m)(l+m+1)}\ \delta_{n\,m+1}$$

$$(L_-)_{nm} = \langle l n | L_- | l m\rangle = \sqrt{(l+m)(l-m+1)}\ \langle l n | l\ m-1\rangle = \sqrt{(l+m)(l-m+1)}\ \delta_{n\,m-1}$$

$l=0 \quad m=0 \quad \vec{L}=0$

$l=\frac{1}{2} \quad m=\frac{1}{2}, -\frac{1}{2}$

$$L_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$$

$$\begin{array}{c|cc} & m=l & m=-l \\ \hline m=l & & \\ m=-l & & \end{array}$$

$$(L_+)_{m+1\,m} = \sqrt{(l-m)(l+m+1)}$$

$L_{\frac{1}{2}\frac{1}{2}} = 0 \quad (L_+)_{-\frac{1}{2}-\frac{1}{2}} = 0$

$L_{\frac{1}{2}-\frac{1}{2}} = \sqrt{\left(\frac{1}{2}-(-\frac{1}{2})\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} = 1$

$L_{-\frac{1}{2}\frac{1}{2}} = 0$

$$L^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\cdot I^{\leq E}$$

$$L_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$L_+ = L_x + i L_y$

$L_- = L_x - i L_y$

$(L_-)_{m-1\,m} = \sqrt{(l+m)(l-m+1)}$

$(L_-)_{-\frac{1}{2}\frac{1}{2}} = (L_-)_{-\frac{1}{2}-\frac{1}{2}} = 0 \quad (L_-)_{+\frac{1}{2}-\frac{1}{2}} = 0$

$(L_-)_{-\frac{1}{2}+\frac{1}{2}} = \sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)} = 1$

$$L_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$L_x = \frac{1}{2}(L_+ + L_-) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (\sigma_x)$$

$$L_y = \frac{1}{2i}(L_+ - L_-) = \frac{1}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (\sigma_y)$$

i konačno:

$$\vec{M} = \hbar \vec{L} = \frac{\hbar}{2}\vec{\sigma}$$

Paulijeva matrica

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

→ prostor je 2-dim.

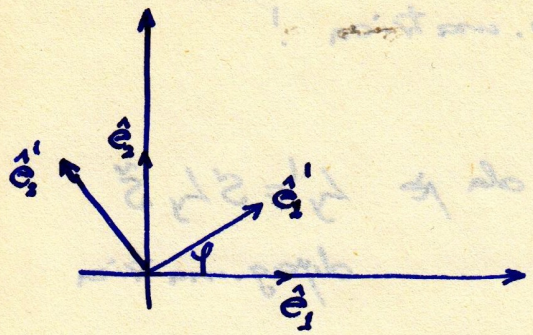
$$\left.\begin{aligned} \hat{e}_1 &= |\tfrac{1}{2}\ \tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \hat{e}_2 &= |\tfrac{1}{2}\ -\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \text{(*) vektori baze „spinskog prostora"}$$

$$L_z |\overset{l}{\tfrac{1}{2}}\ \overset{m}{\tfrac{1}{2}}\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\overset{m\ \ |l\,m\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \frac{1}{2}|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

DZ: Nađi matrice mom. impulsa u slučaju $l=1$

Lang. mom.

→ (3×3) matrica



realna transformacija:
(rotacija)

tj. uzmimo bazu u spinskom 2-dim. prostoru
različitu od (*)

→ matrice angul. mom. u $(\hat{e}_1', \hat{e}_2')$

$$\hat{e}_1' = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \qquad \hat{e}_2' = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

$$\hat{e}_1' = R_\varphi \hat{e}_1 \qquad \Updownarrow \qquad \hat{e}_2' = R_\varphi \hat{e}_2$$

pretp.

$$R_\varphi = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$u\vec{x} \cdot u\vec{y} = \vec{x} \cdot \vec{y}$

i.j. $\hat{e}_1' = \cos\varphi\, \hat{e}_1 + \sin\varphi\, \hat{e}_2$
$\hat{e}_2' = -\sin\varphi\, \hat{e}_1 + \cos\varphi\, \hat{e}_2$ $\Big\} \Rightarrow (\hat{e}_1', \hat{e}_2') = (\hat{e}_1, \hat{e}_2) \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$\hat{e}_1 \cdot \hat{e}_2 = 0$
$\hat{e}_1' \cdot \hat{e}_2' = 0$ $\Big\} \Rightarrow R_\varphi$ unit. matrica
tj. ortogonalna jer je realna

$$\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$R_\varphi = T^\perp$

$T = R_\varphi^\perp = R_\varphi^{-1}$
$T^{-1} = R_\varphi$

R_φ - ortog. matrica

$R_\varphi \cdot R_\varphi^\perp = E$

$R_\varphi^{-1} = R_\varphi^\perp$

vektor u novom sustavu — $X' = A \cdot X$ $\quad A = (T^{-1})^\perp = R_\varphi^\perp$
$(\ ) = R_\varphi^\perp (\ )$

$X' = R_\varphi^\perp X$ $\quad X \xrightarrow{B} (x_1, x_2)$
$\Downarrow$ $\quad \| \quad X' \xrightarrow{B'} (x_1', x_2')$
$= U$

→ inverzna transformacija:

$|\frac{1}{2}\ \frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{e}_1 = R_\varphi^{-1} \hat{e}_1'$ $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{e}_2 = R_\varphi^{-1} \cdot \hat{e}_2'$

$\langle \frac{1}{2}\ \frac{1}{2}| = (1\ 0) = \hat{e}_1^\perp$ $\quad = |\frac{1}{2} -\frac{1}{2}\rangle$

$A' = R_\varphi^\perp A (R_\varphi^\perp)^{-1} = R_\varphi^\perp A R_\varphi$

$$(L_i)_{\frac{1}{2}\frac{1}{2}} = \langle \overset{l\ \frac{1}{2}}{\frac{1}{2}\ \frac{1}{2}} | L_i | \overset{l\ \frac{1}{2}}{\frac{1}{2}\ \frac{1}{2}} \rangle = \hat{e}_1^\perp (L_i) \hat{e}_1 = (R_\varphi^\perp \hat{e}_1')^\perp (L_i)(R_\varphi^\perp \hat{e}_1') =$$

$$= \hat{e}_1'^\perp \underbrace{R_\varphi (L_i) R_\varphi^\perp}_{\cancel{L_i'}} \hat{e}_1'$$

korekcija

$$\Rightarrow \quad L_i' = R_\varphi^\perp L_i R_\varphi^{\cancel{\perp}}$$

- matrica u odnosu na novu bazu

$$L_z' = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ -\sin\varphi & -\cos\varphi \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ -\sin 2\varphi & -\cos 2\varphi \end{pmatrix}$$

dijag. samo za $\varphi = 0$

za $\varphi \neq k\pi$ $\quad L_z'$ nije dijag. matrica

DZ: 1) Za koji je kut φ L_x' dijagon. matrica!

DZ: 2) Naći takvu matricu transf. S da je $L_y' = S L_y \tilde{S}$ dijag. matrica

$U L U^{-1}$ U-unitarna $U \cdot U^{\dagger} = 1$

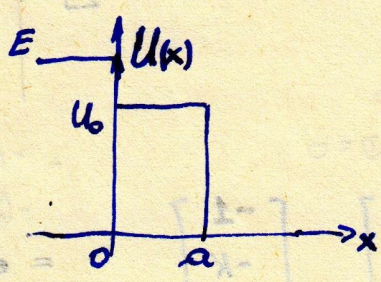
općenito: $L' = S L S^{-1}$

S - unitarna matrica tj. $S\tilde{S} = \tilde{S}S = E$ $\tilde{S} = (S^{\perp})^{*}$

$\Rightarrow L' = S L \tilde{S}$

1) Čestica energije E nailazi na pravokutnu barijeru visine U, širine a. Naći koef. reflex. R, transmisije T za $E \gtrless U_0$

(a) $E > U_0$



$$u''(x) + \frac{2m}{\hbar^2}(E - U(x))u(x) = 0$$

Područja:

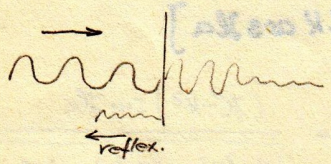
1° $x<0$ $\quad U(x)=0$

$$u''(x) + k^2 u(x) = 0 \qquad k = \frac{1}{\hbar}\sqrt{2mE} \quad \text{valni vektor}$$

$$u(x) = \underbrace{\overset{A}{e^{ikx}}}_{\text{upadni val}} + \overset{B}{A} e^{-ikx}$$

$A' = 1$ iz 4 uvjeta nebi bio ionako odredeno svih 5 koef. (tek iz normiranja)

jer za dobiv R dijelimo sa A'



– mi smo, amplitudu upadnog vala možemo proizv. odabrati

2° $0<x<a$

$$u''(x) + \mathcal{K}^2 u(x) = 0 \qquad \mathcal{K} = \frac{1}{\hbar}\sqrt{2m(E-U_0)}$$

$$u(x) = Be^{i\mathcal{K}x} + Ce^{-i\mathcal{K}x}$$

3° $x>a$ $\quad u''(x) + k^2 u(x) = 0$

$$u(x) = De^{ikx}$$

iza brijega nema, naravno, vala koji bi se širio u protivnom smjeru

reflektirani val javlja se samo ispred potencijalnog brijega

što smo opisali sa Ae^{-ikx}

neprekinutost $u(x)$ & $u'(x)$ na granici

$x=0$ $\quad 1 + A = B + C$

$$ik(1-A) = i\mathcal{K}(B-C)$$

$x=a \qquad Be^{i\mathcal{K}a} + Ce^{-i\mathcal{K}a} = De^{ika}$

$$i\mathcal{K}\left(Be^{i\mathcal{K}a} - Ce^{-i\mathcal{K}a}\right) = ikDe^{ika}$$

$$1\cdot A - B - C + 0\cdot D = -1$$
$$-k\cdot A - \mathcal{K}\cdot B + \mathcal{K}\cdot C + 0\cdot D = -k$$
$$0\cdot A + e^{i\mathcal{K}a}\cdot B + e^{-i\mathcal{K}a}\cdot C - e^{ika}\cdot D = 0$$
$$0\cdot A + \mathcal{K}e^{i\mathcal{K}a}\cdot B - \mathcal{K}\cdot e^{-i\mathcal{K}a}\cdot C - ke^{ika}\cdot D = 0$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -k & -\mathcal{K} & \mathcal{K} & 0 \\ 0 & e^{i\mathcal{K}a} & e^{-i\mathcal{K}a} & -e^{ika} \\ 0 & \mathcal{K}e^{i\mathcal{K}a} & -\mathcal{K}e^{-i\mathcal{K}a} & -ke^{ika} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -1 \\ -k \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -k & -(\mathcal{K}+k) & \mathcal{K}-k & 0 \\ 0 & e^{i\mathcal{K}a} & e^{-i\mathcal{K}a} & -e^{ika} \\ 0 & \mathcal{K}e^{i\mathcal{K}a} & -\mathcal{K}e^{-i\mathcal{K}a} & -ke^{ika} \end{vmatrix} =$$

$$= -(\mathcal{K}+k)\left[-ke^{-i\mathcal{K}a+ika} - \mathcal{K}e^{ika-i\mathcal{K}a}\right]$$
$$-(\mathcal{K}-k)\left[-ke^{i\mathcal{K}a+ika} + \mathcal{K}e^{i\mathcal{K}a+ika}\right] =$$
$$= e^{ika}\left[(\mathcal{K}+k)^2 e^{-i\mathcal{K}a} - (\mathcal{K}-k)^2 e^{i\mathcal{K}a}\right]$$

$$\Delta_1 = (\mathcal{K}^2-k^2)\, e^{ika+i\mathcal{K}a} + (\mathcal{K}^2-k^2)\, e^{ika-i\mathcal{K}a}$$
$$= 2i\,(\mathcal{K}^2-k^2)\, e^{ika}\cdot \sin\mathcal{K}a$$

$$R = \frac{|j_r|}{j_u} \qquad u_u = e^{ikx} \quad j_u = \frac{\hbar k}{m} \qquad u_r = Ae^{-ikx} \quad j_r = -\frac{\hbar k}{m}|A|^2 \quad \Rightarrow R = \frac{\frac{\hbar k}{m}|A|^2}{\frac{\hbar k}{m}}$$

možemo:

~~def~~: $R = \dfrac{\frac{\hbar}{m}k|A|^2}{\frac{\hbar}{m}k} = |A|^2$ ovaj gustoća fluxa

$$j = \rho\cdot v = |\psi|^2\cdot v = |\psi|^2\cdot\frac{\hbar k}{m}$$

$$A = \frac{\Delta_1}{\Delta} = \frac{2i(\mathcal{K}^2-k^2)\cancel{e^{ika}}\sin\mathcal{K}a}{\cancel{e^{ika}}\left[(\mathcal{K}^2+k^2)(-2i)\sin\mathcal{K}a + 4\mathcal{K}k\cos\mathcal{K}a\right]}$$

$$|A|^2 = \frac{2(\mathcal{K}^2-k^2)\sin\mathcal{K}a}{\sqrt{16\mathcal{K}^2k^2\cos^2\mathcal{K}a + (\mathcal{K}^2+k^2)^2\cdot 4\sin^2\mathcal{K}a}} = \frac{(\mathcal{K}^2-k^2)\sin\mathcal{K}a}{\sqrt{4\mathcal{K}^2k^2 - 4\mathcal{K}^2k^2\sin^2\mathcal{K}a + (\mathcal{K}^2+k^2)^2\sin^2\mathcal{K}a}}$$
$$= \frac{(\mathcal{K}^2-k^2)\sin\mathcal{K}a}{\sqrt{4\mathcal{K}^2k^2 + (\mathcal{K}^2-k^2)^2\sin^2\mathcal{K}a}}$$

$$R = |A|^2 = \left[1 + \frac{4\mathcal{K}^2k^2}{(\mathcal{K}^2-k^2)^2\sin^2\mathcal{K}a}\right]^{-1}$$

$R=0$ za $\mathcal{K}=k$ tj. $U_0=0$

$R=1$ za $\mathcal{K}=0$ tj. $E=U_0$

$u_t = De^{ikx}$

~~po def~~: $T = \dfrac{j_t}{j_u} = \dfrac{\frac{\hbar k}{m}|D|^2}{\frac{\hbar k}{m}}$

$T = |D|^2$ ali zbog $R+T=1$ (*)

možemo napisati bez odred. D — odredivanjem D bi potvrdili (*)

$$P = T = 1 - R = \frac{4\mathcal{K}^2k^2}{4\mathcal{K}^2k^2 + (\mathcal{K}^2-k^2)^2\sin^2\mathcal{K}a} = \left[1 + \frac{(\mathcal{K}^2-k^2)^2\sin^2\mathcal{K}a}{4\mathcal{K}^2k^2}\right]^{-1}$$

za $E > U_0$

→ diskusija:

$P = T = 1$

max. transmisija → $\sin \mathcal{K}a = 0$ ovisi o energiji čestice

$\Rightarrow T = 1 \quad R = 0$

$\Rightarrow \mathcal{K}a = n\pi \qquad n = 1, 2, \ldots$ → i to samo za odred. energije - prolaz kao da nema barijere

$$\mathcal{K}^2 = \frac{n^2\pi^2}{a^2} = \frac{1}{\hbar^2}(2m)(E - U_0)$$

$$\Rightarrow E_n = \underbrace{\frac{\hbar^2\pi^2}{2ma^2} n^2}_{E_{K_n}} + U_0 \qquad n = 1, 2, 3, \ldots$$

čestica neće ospetiti pot. brijeg

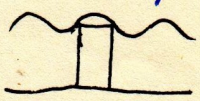
$$E_{K_n} = \frac{\hbar^2\pi^2}{2ma^2} n^2 = \frac{h^2}{2ma^2}\frac{n^2}{4}$$

$$E_K = \frac{\hbar^2 K^2}{2m} = \frac{\hbar^2 \cdot (2\pi)^2}{2m\lambda} = \frac{h^2}{2m\lambda^2}$$

$$\frac{1}{\lambda^2} = \frac{n^2}{4a^2}$$

$$\Rightarrow a = \frac{n}{2}\lambda$$

širina za koju čestica neće ospetiti barijeru



max. refleksija ($T = 0 \quad R = 1$ tj. potpuna reflek. za $E > U_0$ ne ∃)
vec' za $E = U_0$ ∃

$|\sin \mathcal{K}a| = 1 \Rightarrow \mathcal{K}a = (2n+1)\frac{\pi}{2}$

$$\Rightarrow a = (2n+1)\frac{\lambda}{4}$$

širina za max. reflek.

(b) $E < U_0$

$1^\circ \quad u''(x) + k^2 u(x) = 0 \qquad k = \frac{1}{\hbar}\sqrt{2mE}$

$2^\circ \quad u''(x) + \frac{2m}{\hbar^2}(E-U_0) = 0$

$u''(x) + (-1)\underbrace{\frac{2m}{\hbar^2}(U_0-E)}_{k_0^2} = 0 \qquad k_0 = \frac{1}{\hbar}\sqrt{2m(U_0-E)}$

$\Rightarrow u(x) = B e^{k_0 x} + C e^{-k_0 x}$

$x=0 \quad 1 + A = B + C$

$ik(1-A) = k_0(B-C) \Rightarrow k(1-A) = \frac{k_0}{i}(B-C)$ **(*)**

odgovara prijelazu sustava $\mathcal{H}$

$\mathcal{H}^2 = \frac{2m}{\hbar^2}(E-U_0)$

$k_0^2 = \frac{2m}{\hbar^2}(U_0-E)$

$\overset{(*)}{\Rightarrow} \mathcal{H} = -ik_0$

$\mathcal{H} = -ik_0$

$\sin \mathcal{H}a = \sin(-iak_0) = \frac{e^{i(-iak_0)} - e^{-i(-iak_0)}}{2i} = \frac{1}{2i}\left(e^{ak_0} - e^{-ak_0}\right) = -i\,\mathrm{sh}\,ak_0$

$(\mathcal{H}^2 - k^2) = (-k_0^2 - k^2)$

umjesto rješavanja supstituiramo!

$$\Rightarrow R = \left[1 + \frac{4k_0^2k^2}{(k_0^2+k^2)^2\,\mathrm{sh}^2 k_0 a}\right]^{-1}$$

$$T = \left[1 + \frac{(k_0^2+k^2)^2\,\mathrm{sh}^2 k_0 a}{4k_0^2k^2}\right]^{-1}$$

→ Klasični limes:

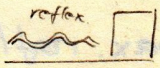
Q.M. → Klas. meh. za $\hbar \to 0$

$\hbar \to 0 \Rightarrow k, k_0 \to \infty$

$\mathrm{sh}\, k_0 a \to \infty \Rightarrow T \to 0$

jer $e^{k_0 a} \to \infty$ najbrže

u klas. fizici za $E < U_0$ čestica ne može proći



→ Quasi-Klas. limes:

$k_0 a \gg 1$ kao $la \gg 1$ na (22)

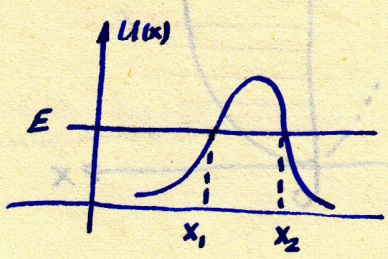
$\Rightarrow \mathrm{sh}\, k_0 a \approx \frac{1}{2} e^{k_0 a}$

$T \approx \frac{4k_0^2k^2}{4k_0^2k^2 + \underbrace{\frac{1}{4}(k_0^2+k^2)^2 e^{2k_0 a}}_{\text{zanemarivo}}} \approx \frac{16k_0^2k^2}{(k_0^2+k^2)^2}$

$$T \approx \frac{16k^2k_0^2}{(k^2+k_0^2)^2} e^{-2k_0 a} = \frac{16E(U_0-E)}{U_0^2} e^{-\frac{2a}{\hbar}\sqrt{2m(U_0-E)}}$$

transmisija vrlo brzo opada sa a, U_0 & m

→ na osnovu tog rezultata se T približno računa za slučaj

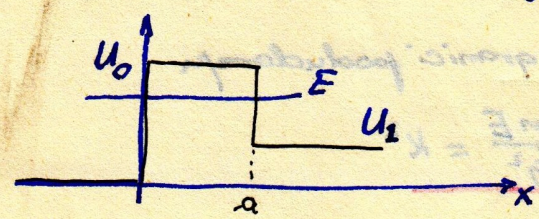


$$P = T \sim e^{-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(U(x)-E)}\,dx}$$

za $la \gg 1$ $\quad l = \frac{1}{\hbar}\sqrt{2m(U_0-E)}$

Iz:

1) Izračunati koef. transmisije T za

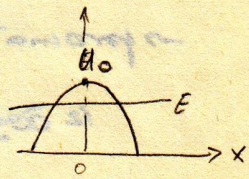


ograničeno za $U_1 < E < U_0$

2) Izračunati preko približne formule T za

153. str.

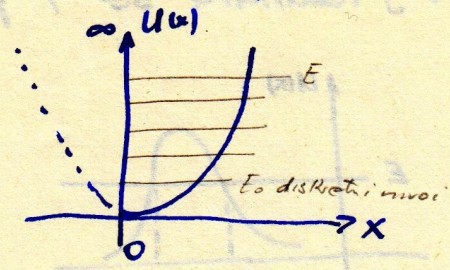
$$U(x) = \begin{cases} U_0\left(1-\frac{x^2}{a^2}\right) & |x|<a \\ 0 & |x|>a \end{cases} \qquad E<U_0$$



1) Riješiti problem gibanja čestice u potencijalu

$$U(x) = \begin{cases} \infty & x<0 \\ \frac{\alpha x^2}{2} & x>0 \end{cases}$$

sliono H.O.



$\alpha > 0$

$x>0 \qquad u''(x) + \frac{2m}{\hbar^2}\left(E - \frac{\alpha x^2}{2}\right)u(x) = 0$

$x<0 \qquad u(x) = 0$ jer ako $u(x) \neq 0 \Rightarrow u''(x) \to \infty \Rightarrow u(x) = \infty$ neregularno

i ta dva rješenja se na granici podudaraju

uvodimo: $a^4 = \frac{m\alpha}{\hbar^2} \qquad \frac{2mE}{\hbar^2} = k^2$

$\Rightarrow$

$$u''(x) + k^2 u(x) - a^4 x^2 u(x) = 0 \quad *)$$

- u nuli jba. nema singularitet

→ promatramo asimptotsko ponašanje $x \to \infty$, $x \to 0$

iz čega zaključujemo pravi oblik $u(x)$

$x \to \infty \Rightarrow u''(x) - a^4 x^2 u(x) = 0$

→ pretp. rješ: $u(x) \sim e^{-\frac{a^2x^2}{2}}$ *) $\qquad e^{\frac{a^2x^2}{2}}$ ne fizik. smisla za $x \to \infty$

$$u'(x) \sim -a^2 x\, e^{-\frac{a^2x^2}{2}} = -a^2 x\, u(x)$$

$$u''(x) \sim \underbrace{-a^2 e^{-\frac{a^2x^2}{2}}}_{\text{zanemarivo za } x\to\infty} + a^4 x^2 e^{-\frac{a^2x^2}{2}} \approx a^4 x^2 u(x)$$

*) egzaktno rješ. za $k^2 = a^2$ jbe. *)

→ pretp. da je prava valna fja. tj. za $\forall x$

$$u(x) = e^{-\frac{a^2x^2}{2}} P(x)$$

$u(x) = e^{-\frac{a^2x^2}{2}} P(x)$ $\qquad u''(x) + (k^2 - a^4x^2)\, u(x) = 0$

$$u'(x) = -a^2 x\, e^{-\frac{a^2x^2}{2}} P(x) + e^{-\frac{a^2x^2}{2}} P'(x)$$

$$u''(x) = -a^2 e^{-\frac{a^2x^2}{2}} P(x) + a^4x^2 e^{-\frac{a^2x^2}{2}} P(x) - a^2 x\, e^{-\frac{a^2x^2}{2}} P'(x) - a^2 x\, e^{-\frac{a^2x^2}{2}} P'(x) + e^{-\frac{a^2x^2}{2}} P''(x)$$

$$u''(x) = e^{-\frac{a^2x^2}{2}} \left[(-a^2 + a^4x^2) P(x) - 2a^2 x P'(x) + P''(x) \right] + e^{-\frac{a^2x^2}{2}} \left[(k^2 - a^4x^2) P(x) \right] \qquad / : e^{-\frac{a^2x^2}{2}} \neq 0 \text{ za } x < \infty$$

$$\Rightarrow \quad P''(x) - \underbrace{2a^2 x}_{-2a^2 \frac{z}{a}} P'(x) + (k^2 - a^2) P(x) = 0$$

supst. $z = ax$ $\qquad \Rightarrow \frac{d^2}{dx^2} = a^2 \frac{d^2}{dz^2}$ $\qquad \frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz}$

$P(x) = P(\frac{z}{a}) = P_1(z)$

$$a^2 P_1''(z) - 2a^2 \cdot \frac{z}{a} \cdot a P_1'(z) + (k^2 - a^2) P_1(z) = 0 \qquad / : a^2 \neq 0$$

$$\Rightarrow \quad P_1''(z) - 2z P_1'(z) + \left(\frac{k^2}{a^2} - 1\right) P_1(z) = 0$$

bio je označen k

- diskusija kod H.O. str 29 $\qquad P_1(z) = \sum_{m=0}^{n} C_m s^m \Rightarrow$ n-konačan

& isto za Hermit-ove polinome

$$H_n''(s) - 2s H_n'(s) + 2n H_n(s) = 0$$

$$\Rightarrow \frac{k^2}{a^2} - 1 = 2n \qquad n = 0, 1, 2, \ldots \qquad \Rightarrow P_{1n}(z) = H_n(z)$$

$$\Rightarrow k_n^2 = (2n+1) a^2$$

$$E_n = \frac{\hbar^2}{2m} \cdot (2n+1) \cdot \sqrt{\frac{m\alpha}{\hbar^2}} = \frac{\hbar}{2} \sqrt{\frac{\alpha}{m}} (2n+1) \qquad n = 0, 1, 2, \ldots$$

$$\Rightarrow u_n(x) = A_n e^{-\frac{a^2x^2}{2}} H_n(ax)$$

↓normiranje

- za istu energiju samo jedan valna fja.

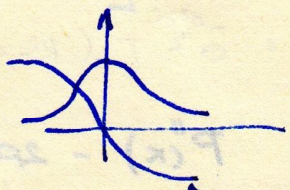
$\Rightarrow g_n = 1$

$$\int |\psi|^2 d\tau = 1 \quad \Rightarrow \int_{-\infty}^{\infty} u_n^2(x)\, dx = 1$$

rubni uvjeti

$u_n(0)=0$ i) → dati će uvjet na $\underline{n}$

općenito Hermiet-ovi polinomi $\quad H_n(x) = (-1)^n H_n(-x)$



neparni Hermiet-ov polinom

i) ⇒ $H_n(0)=0 \Rightarrow n=1,3,5,\dots$

$$n = 2p+1 \qquad p=0,1,2,3,\dots$$

$u_n(x) = u_{2p+1}(x) = u_{1p}(x)$

$$\Rightarrow \quad u_{1p}(x) = A'_p\, e^{-\frac{a^2x^2}{2}} H_{2p+1}(ax) \qquad p=0,1,2,\dots$$

$E_n = \frac{\hbar}{2}\sqrt{\frac{\alpha}{m}}\,(2n+1)$

$$E_{1p} = \frac{\hbar}{2}\underbrace{\sqrt{\frac{\alpha}{m}}}_{=\omega}\,(4p+3) \qquad p=0,1,2,\dots$$

$\frac{\alpha x^2}{2} \quad \alpha = m\omega^2$

- zbog prisilnog potencijalnog kvadrata E se podigla za $\hbar\omega$

$$E_0 = \frac{3}{2}\hbar\omega \quad \text{, dok inače } E_0 = \frac{\hbar\omega}{2}$$

$s = ax$
$ds = a\,dx \Rightarrow dx = \frac{1}{a}ds$

II: 1) Odrediti A_p' - na str 30 $\quad A_p' = A_{2p+1} = \frac{1}{\sqrt{2^{2p+1}\cdot(2p+1)!\,a^{-1}\sqrt{\pi}}}$

$$1 = \int_0^\infty u_p^2(x)\,dx$$

$\frac{1}{a}ds$

i pokazati da Hermiet-ovi polinomi zadovoljavaju uvjetu ortonormiranosti

$$\int_{-\infty}^{\infty} H_n(x)\,H_m(x)\,e^{-x^2}dx = \delta_{mn}\sqrt{\pi}\cdot 2^n\cdot n!$$

trick!!! iz fg. izvodnice

$$\phi(x,z) = e^{2xz-z^2} = \sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x) \qquad \text{po } z \text{ u Tayl. red. u okolini nule}$$

Gauss-ov integral $\quad 1^\circ$

$$f(\lambda,z) = \int_{-\infty}^{\infty} e^{-x^2}\phi(x,\lambda)\,\phi(x,z)\,dx = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^n}{n!}\frac{\lambda^m}{m!}\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)\,dx$$

$-(x^2+2xz+z^2+2z\lambda-2x\lambda+\lambda^2)$

$-2z\lambda \quad +2z\lambda$

2°

$$f(\lambda,z)=\int_{-\infty}^{\infty} e^{-x^2}\cdot e^{2xz-z^2}\cdot e^{2x\lambda-\lambda^2}\,dx = e^{2z\lambda}\cdot\int_{-\infty}^{\infty} e^{-(x-z-\lambda)^2}\,d(x-z-\lambda)=\sqrt{\pi}\,e^{2z\cdot\lambda}$$

$$f(\lambda,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^n z^m}{n!\,m!}\left(\frac{d^n}{d\lambda^n}\frac{d^m}{dz^m}f(\lambda,z)\right)\Bigg|_{z=\lambda=0} =$$

Taylor-ov red. za 2 varijable

$$\left[\frac{d^n}{d\lambda^n}\left(\sqrt{\pi}\,(2\lambda)^m e^{2z\lambda}\right)\right]_{z=\lambda=0}\neq 0 \text{ za } n=m \text{ i to } =\sqrt{\pi}\cdot 2^m\cdot m!$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^n z^m}{n!\,m!}\cdot\sqrt{\pi}\cdot 2^m\cdot m!\,\delta_{nm}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2z)^m\cdot\lambda^n}{n!}\sqrt{\pi}\cdot\delta_{nm}$$

gdje $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}$ pokazi isto kao $\sum_{n=0}^{\infty}\frac{1}{n!}\left[\lambda f(\lambda,z)_\lambda + z f(\lambda,z)_z\right]^{(n)}$

potovara deriv. i faktor

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(2z\cdot\lambda)^n}{n!}\cdot\sqrt{\pi}\,\delta_{nm}$$

1° & 2° ⇒ $$\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)\,dx = n!\,2^n\sqrt{\pi}\,\delta_{nm}$$

$p_k=-i\hbar\frac{\partial}{\partial x_k}$

II z. 2) Izračunati u sferonom sustavu

a) $M_x=i\hbar\left(\sin\varphi\frac{\partial}{\partial\vartheta}+\operatorname{ctg}\vartheta\cos\varphi\frac{\partial}{\partial\varphi}\right)$

$M_y=-i\hbar\left(+\cos\varphi\frac{\partial}{\partial\vartheta}-\operatorname{ctg}\vartheta\sin\varphi\frac{\partial}{\partial\varphi}\right)$

$M_z=-i\hbar\frac{\partial}{\partial\varphi}$

b) $\Delta=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ \ )-\frac{\vec{M}^2}{\hbar^2 r^2}$ $\qquad \vec{M}^2=M_x^2+M_y^2+M_z^2$

a) $M_x=y p_z-z p_y \rightarrow \hat{M}_x=-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)$

$x=r\sin\vartheta\cos\varphi$
$y=r\sin\vartheta\sin\varphi$
$z=r\cos\vartheta$

$$\frac{\partial}{\partial z}=\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\vartheta}{\partial z}\frac{\partial}{\partial\vartheta}+\underbrace{\frac{\partial\varphi}{\partial z}}_{=0}\frac{\partial}{\partial\varphi}$$

i) $r=\sqrt{x^2+y^2+z^2}$

ii) $\cos\vartheta=\frac{z}{\sqrt{x^2+y^2+z^2}}=\frac{z}{r}$

iii) $\operatorname{tg}\varphi=\frac{y}{x}$

i) ⇒ $dr=\frac{x}{r}dx+\frac{y}{r}dy+\frac{z}{r}dz$

$dr=\sin\vartheta\cos\varphi\,dx+\sin\vartheta\sin\varphi\,dy+\cos\vartheta\,dz$

$$d\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = -\frac{1}{2}\frac{2x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}\,dx + \dots$$

$$\frac{t}{d} = \cos\vartheta = \frac{z}{\sqrt{\ }} = \frac{z}{r} \qquad d(\cos\vartheta) = \frac{1}{r}dz + z\cdot d\left(\frac{1}{r}\right)$$

$$-\frac{1}{r^2}dr = -\frac{r\,dr}{r^3}$$

ii) $\Rightarrow$ $$-\sin\vartheta\, d\vartheta = \frac{dz}{r} + z\cdot\left(-\frac{x}{r^3}dx - \frac{y}{r^3}dy - \frac{z}{r^3}dz\right)$$

$x = r\sin\vartheta\cos\varphi$
$y = r\sin\vartheta\sin\varphi$
$z = r\cos\vartheta$.

$$d\vartheta = -\frac{1}{\sin\vartheta}\left[-\frac{z\cdot x}{r^3}dx - \frac{z\cdot y}{r^3}dy + \underbrace{\left(-\frac{z^2}{r^3}+\frac{1}{r}\right)}_{\frac{-\cos^2\vartheta+1}{r}}dz\right]$$

$$d\vartheta = -\frac{1}{\sin\vartheta}\left[-\frac{\cos\vartheta\sin\vartheta\cos\varphi}{r}dx - \frac{\cos\vartheta\sin\vartheta\sin\varphi}{r}dy + \frac{1-\cos^2\vartheta}{r}dz\right]$$

$$d\vartheta = \frac{\cos\vartheta\cos\varphi}{r}dx + \frac{\cos\vartheta\sin\varphi}{r}dy - \frac{\sin\vartheta}{r}dz$$

iii) $\Rightarrow$ $$\frac{d\varphi}{\cos^2\varphi} = \frac{dy}{x} - \frac{y\,dx}{x^2} = \underbrace{\frac{1}{r\sin\vartheta\cos\varphi}}_{\frac{\partial\varphi}{\partial y}/\cos^2\varphi}dy - \underbrace{\frac{\sin\varphi}{r\sin\vartheta\cos^2\varphi}}_{\frac{\partial\varphi}{\partial x}/\cos^2\varphi}dx$$

$$d\varphi = \frac{\cos\varphi}{r\sin\vartheta}dy - \frac{\sin\varphi}{r\sin\vartheta}dx$$

$$\Rightarrow \frac{\partial}{\partial z} = \cos\vartheta\frac{\partial}{\partial r} + \left(-\frac{\sin\vartheta}{r}\right)\frac{\partial}{\partial\vartheta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\vartheta}{\partial y}\frac{\partial}{\partial\vartheta} + \frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}$$

$$= \sin\vartheta\sin\varphi\frac{\partial}{\partial r} + \left(\frac{\cos\vartheta\sin\varphi}{r}\right)\frac{\partial}{\partial\vartheta} + \frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi}$$

$$M_x = -i\hbar\left[r\sin\vartheta\sin\varphi\left(\cos\vartheta\frac{\partial}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial}{\partial\vartheta}\right) - r\cos\vartheta\left(\sin\vartheta\sin\varphi\frac{\partial}{\partial r} + \frac{\cos\vartheta\sin\varphi}{r}\frac{\partial}{\partial\vartheta} + \frac{\cos\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\right]$$

$$= -i\hbar\left[\left(r\sin\vartheta\sin\varphi\cos\vartheta - \cancel{r\cos\vartheta\sin\vartheta\sin\varphi}\right)\frac{\partial}{\partial r} + \left(-\sin^2\vartheta\sin\varphi - \cancel{\cos^2\vartheta\sin\varphi}\right)\frac{\partial}{\partial\vartheta} + \left(-\cos\varphi\frac{\cos\vartheta}{\sin\vartheta}\right)\frac{\partial}{\partial\varphi}\right]$$

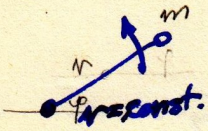
$$M_x = i\hbar\left[\sin\varphi\frac{\partial}{\partial\vartheta} + \cos\varphi\cot\vartheta\frac{\partial}{\partial\varphi}\right]$$

$$\Lambda = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}$$

Ravninski rotator

– slobodni $U=0$

1 st. slobode



$\Downarrow$

$J = mr^2$

$U=0$ $\quad T = \underset{=0}{T_{transl.}} + T_{rot}$

$$E = H = T + U = T = \frac{M^2}{2J}$$

$$H = \frac{\vec{M}^2}{2J}$$

– ravninski rotator $\quad \vec{M} = (0,0,M_z)$ $\Big\} \Rightarrow H = \frac{M_z^2}{2J}$

$$\hat{H}u(\varphi) = E\cdot u(\varphi) \qquad \hat{M}_z^2 = -\hbar^2\frac{\partial^2}{\partial\varphi^2}$$

$$\left(-\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\varphi^2} - E\right)u(\varphi) = 0 \quad \Longleftrightarrow$$

– ilustracija tipičnih pitanja:

$$\frac{\hat{M}_z^2}{2J}u(\varphi) = E\,u(\varphi) \Rightarrow \hat{M}_z^2 u(\varphi) = (2JE)\,u(\varphi)$$

1) problem se svodi na to da se nađe vlast. fja. operatora $\hat{M}_z$
– to je rješ. jbe. gibanja

$u(\varphi)$ vlast. fja. za operator $\hat{M}_z^2$, pa će biti i za operator $\hat{M}_z$

u polarnom sustavu:

$$\hat{M}_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$$

$$**)\quad \hat{M}_z u(\varphi) = \underbrace{\hbar m}_{\text{vlastita vrijednost}}\, u(\varphi)$$

nepoznata konst.

– ne možemo reći da je cijeli broj i kojim skupom može ići: konačnim (kao u slučaju $U(r)$ $m=\pm l,\dots 0$) ili beskonačnim

$$\frac{1}{i}\hbar\frac{\partial u(\varphi)}{\partial\varphi} = \hbar m\, u(\varphi)$$

$$\Rightarrow u(\varphi) = A e^{im\varphi}$$

$$e^{im\varphi} = e^{im(\varphi+2\pi)} \quad \text{jednoznačnost} \Rightarrow m = 0, \pm1, \pm2, \dots \quad \text{tj. } m\in\mathbb{Z}$$

$\exists$ ∞ mnogo vlastitih fja.

$$u_m(\varphi) = A_m e^{im\varphi} \qquad \int|\psi|^2 d\tau = 1 \Rightarrow A_m^2\int_0^{2\pi}\underbrace{|e^{im\varphi}|^2}_{=1}d\varphi = 1 \qquad A_m^2\cdot 2\pi = 1$$

$$\Rightarrow u_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}$$

Schröd. jba. (*) $\frac{M_z^2}{2J} \cdot u(\varphi) = E u(\varphi)$

$$\frac{\hat{M}_z^2}{2J} u_m(\varphi) = E_m u_m(\varphi) \qquad \hat{M}_z u_m(\varphi) = \hbar m u_m(\varphi)$$

**) $\Rightarrow$ $\frac{\hbar^2 m^2}{2J} u_m(\varphi)$, mjenosa

$$\Rightarrow E_m = \frac{\hbar^2 m^2}{2I} \qquad m = 0, \pm 1, \pm 2, \dots$$

- tosu moguća stanja ravninskog rotatora
- u fizici nikad nemamo čista stanja, već vjerojatnosti za pojedina stanja

$u_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$ $m \in \mathbb{Z}$ ortonormirane valna fje

str. 38 opća rješ. izgrad. iz spektra rješenja

origin. !!! $\psi(\vec{r},t) = \psi(\varphi,t) = \sum_{m=-\infty}^{\infty} C_m u_m(\varphi) e^{-\frac{i}{\hbar}E_m t}$ gdje $E_m = E_{-m}$

iz poč. uvjeta:

$$f(\varphi) = \psi(\varphi,0) = \sum_{m=-\infty}^{\infty} C_m u_m(\varphi)$$

sa str. 19 $\psi(\vec{r},t) = u(\vec{r}) \cdot \varphi(t)$

$\Rightarrow \varphi(t) = C e^{-\frac{i}{\hbar}Et}$

valna fja: $\psi(\vec{r},t)$ za $t=0$ poč. uvjeti

$$f(\varphi) = \sum_{m=-\infty}^{\infty} C_m u_m(\varphi) \quad / \cdot u_{n'}^* \int_0^{2\pi} d\varphi$$

$$\Rightarrow C_m = \int_0^{2\pi} u_m^*(\varphi) f(\varphi) d\varphi$$

$$\rho = |\psi|^2 = \sum_n \sum_m C_n^* C_m e^{i\omega_{nm}t} u_n^* u_m$$

$$\int |\psi|^2 d\tau = 1 \Rightarrow \sum_n |C_n|^2 = 1$$

$$1 = \int |\psi|^2 d\tau = \sum_n \sum_m C_n^* C_m e^{i\omega_{nm}t} \underbrace{\int u_n^* u_m d\tau}_{\delta_{nm}} \overset{\omega_{nn}=0}{=}$$

$$= \sum_n \sum_m C_n^* C_m \int u_n^* u_m d\tau = \int \psi^*(\vec{r},0) \cdot \psi(\vec{r},0) d\tau$$

$$= \int |\psi(\vec{r},0)|^2 d\tau \qquad \psi(\vec{r},0) = g(\vec{r})$$

$$W_m = |C_m|^2$$

vjerojatnost da sustav nalazi m-tom kvantnom stanju s valnom fjom u_m i energijom E_m

$$\sum_{m=-\infty}^{\infty} W_m = 1$$

mogli smo dob. ti i iz **)

$$\left(\frac{\partial^2}{\partial\varphi^2} + \frac{2JE}{\hbar^2}\right) u(\varphi) = 0 \qquad \frac{2JE}{\hbar^2} = k^2$$

$u(\varphi) = e^{r\varphi} \Rightarrow r^2 + k = 0$

$r = \pm i\sqrt{k}$

$u(\varphi) = e^{i\sqrt{k}\varphi}$

$e^{i\sqrt{k}\varphi} = e^{i\sqrt{k}(\varphi+2\pi)}$

$\Rightarrow \sqrt{k} = m$ $m \in \mathbb{Z}$ cijeli broj $\Rightarrow \sqrt{k} \in \mathbb{R}$

$\Rightarrow \boxed{E = \frac{\hbar^2 m^2}{2J}}$

Primjer:

1) Ravninski rotator nalazi se u stanju koje se može opisati valnom fjom.

$$u(\varphi) = A\sin^2\varphi$$

$g(\vec{r}) = \psi(\vec{r},0)$

const. normiranja

$$1 = \int|\psi|^2 d\tau = \int|\psi(\vec{r},0)|^2 d\tau = \int|g(\vec{r})|^2 d\tau = \sum_n |c_n|^2$$

1° - sigurni smo da je u tom stanju

$\frac{3}{8}\varphi - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x$

$$1 = \int_0^{2\pi} u^2(\varphi)\,d\varphi = A^2\int_0^{2\pi}\sin^4\varphi\,d\varphi = A^2\cdot\frac{3}{8}\cdot 2\pi$$

$$\Rightarrow A = \frac{2}{\sqrt{3\pi}}$$

$$u(\varphi) = \frac{2}{\sqrt{3\pi}}\sin^2\varphi$$

?

na z-osi

- čestica opisana tom valnom fjom. može imati neku diskretnost

2° uz tu valnu fju., koje sve vrijednosti može imati moment impulsa (tj. koji m) i s kojom vjerojatnošću

→ prikaz u obliku superpozicije preko vlastitih fja. za taj moment impulsa

$u_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}$

$\cos 2\varphi = 1 - 2\sin^2\varphi$

$$u(\varphi) = \frac{2}{\sqrt{3\pi}}\frac{1}{2}(1-\cos 2\varphi) = \frac{1}{\sqrt{3\pi}}\left[1 - \frac{1}{2}\left(e^{i2\varphi} + e^{-i2\varphi}\right)\right]$$

$$= \sqrt{\frac{2}{3}}\left[\frac{1}{\sqrt{2\pi}} - \frac{1}{2}\frac{1}{\sqrt{2\pi}}e^{i2\varphi} - \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}e^{-i2\varphi}\right]$$

$$= \sqrt{\frac{2}{3}}\,u_0(\varphi) - \frac{1}{2}\sqrt{\frac{2}{3}}\,u_2(\varphi) - \frac{1}{2}\sqrt{\frac{2}{3}}\,u_{-2}(\varphi)$$

vlastite fje. za operator $\hat{M}_z$

Npr. $H = E$

$\hat{H}u = E\cdot u$

$\hat{M}_z u(\varphi) = \hbar m\, u(\varphi)$

$\hat{M}_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$

$\Rightarrow M_z = \hbar m$

drugačim

$$-\frac{1}{2}\sqrt{\frac{2}{3}} = \int_0^{2\pi} u_2^*(\varphi)\,u(\varphi)\,d\varphi = \frac{1}{\sqrt{2\pi}}\frac{2}{\sqrt{3\pi}}\int_0^{2\pi} e^{i2\varphi}\sin^2\varphi\,d\varphi$$

$\Rightarrow M_z$ može biti $\hbar\cdot 0$, $2\hbar$, $-2\hbar$

sa vjerojatnostima s kojima je u prip. stanju

$$W_0 = \frac{2}{3} \qquad W_2 = \frac{1}{6} \qquad W_{-2} = \frac{1}{6}$$

3° Srednja vrijednost $\overline{M_z}$ operatora $\hat{M}$ u stanju opisanom s valnom fjom $u(\varphi) = \frac{2}{\sqrt{3\pi}} \sin^2\varphi$

$\psi(\vec{r},t) = \sum_n c_n u_n e^{-\frac{i}{\hbar}E_n t}$ $\qquad \hat{H} u_n = E_n u_n$ $\qquad$ orijen.

postulat $\overline{G} = \int \psi^* G \psi \, d\tau \overset{\sim}{=} \sum_n \sum_m c_n^* c_m e^{i\omega_{mn} t} G_{nm}$

$G_{nm} \overset{df}{=} \int u_n^* \hat{G} u_m \, d\tau$ $\Big\} \Rightarrow G_{nm} = g_n \delta_{nm} \Rightarrow \overline{G} = \sum_n |c_n|^2 g_n$

ako još $\hat{G} u_n = g_n u_n$ $\qquad \omega_{nn} = 0$ $\qquad$ možemo pisati bez $e^{i\omega_{mn}t}$ *)

$\overline{G} = \sum_n \sum_m c_n^* c_m \int u_n^* \hat{G} u_m \, d\tau = \int u^*(\vec{r}) \hat{G} u(\vec{r}) \, d\tau$ **)

$\overline{G} = \int \psi^* \hat{G} \psi \, d\tau$, $\psi(\vec{r},t) = u(\vec{r}) \varphi(t)$ $\Big\} \Rightarrow$

$\Rightarrow \overline{G} = \int u^*(\vec{r}) \hat{G} u(\vec{r}) \, d\tau$

$$\overline{M_z} \overset{\text{zbog } **)}{=} \int_0^{2\pi} u^*(\varphi) \cdot \hat{M}_z u(\varphi) \, d\varphi$$

$$\overline{M_z} = \frac{4}{3\pi} \int_0^{2\pi} \sin^2\varphi \cdot \frac{\hbar}{i} \frac{\partial}{\partial\varphi} \sin^2\varphi \, d\varphi = \frac{8\hbar}{3\pi i} \int_0^{2\pi} \sin^3\varphi \cdot \underbrace{\cos\varphi \, d\varphi}_{d(\sin\varphi)} = \frac{8\hbar}{3\pi i} \cdot \frac{1}{4} \cdot \sin^4\varphi \Big|_0^{2\pi}$$

$2\sin\varphi\cos\varphi$

ovdje možemo pr. $\psi(\varphi,t) = \frac{1}{\sqrt{3\pi}} [1 - \cos 2\varphi] e^{-i\frac{2\pi}{T}t}$ $\Big\} \overline{M_z} = 0$

na drugi način:

*) $\Rightarrow \overline{G} = 0 \cdot \hbar \cdot \frac{2}{3} + 2\hbar \cdot \frac{1}{6} + (-2\hbar) \frac{1}{6} = 0$

2) Neka se u $t=0$ čestica opisivala valnom fjom.

$$\psi(\varphi,0) = \frac{2}{\sqrt{3\pi}} \sin^2\varphi$$

Kako se vremenski odvija stanje, odnosno kakav je izgled valne fje $\psi(\varphi,t)$ za $t>0$

- nestacionarna Šhröd. jba.

$$\hat{H} \psi(\vec{r},t) = \hbar i \frac{\partial \psi(\vec{r},t)}{\partial t}$$

$$\hat{H} \psi(\varphi,t) = -\frac{\hbar}{i} \frac{\partial \psi(\varphi,t)}{\partial t}$$

- na osnovu principa superpozicije

$$\psi(\varphi,t) = \sum_m c_m u_m(\varphi) e^{-i\frac{E_m}{\hbar}t}$$

vlast. vrij.

$$\psi(\varphi,0) = \frac{2}{\sqrt{3\pi}} \sin^2\varphi = \sum_m c_m u_m(\varphi)$$

$$\Rightarrow \underline{c_0 = \sqrt{\tfrac{2}{3}}} \qquad \underline{c_2 = c_{-2} = -\sqrt{\tfrac{1}{6}}}$$

C_0, C_2, C_{-2} vrijede za sva vremena $\quad E_m = \frac{\hbar^2 m^2}{2I} \quad \Rightarrow E_0 = 0$

$$\Rightarrow \psi(\varphi, t) = \sqrt{\frac{2}{3}}\, u_0(\varphi) - \sqrt{\frac{1}{6}}\left[u_2(\varphi)\, e^{-i\frac{E_2}{\hbar}t} + u_{-2}(\varphi)\, e^{-i\frac{E_{-2}}{\hbar}t}\right]$$

$$= \sqrt{\frac{2}{3}} \cdot \frac{1}{\sqrt{2\pi}} - \sqrt{\frac{1}{6}} \cdot \left[\frac{1}{\sqrt{2\pi}} \cdot e^{i2\varphi} \cdot e^{-i\frac{\hbar \cdot 2}{I}t} + \frac{1}{\sqrt{2\pi}} e^{-i2\varphi} \cdot e^{-i\frac{2\hbar}{I}t}\right]$$

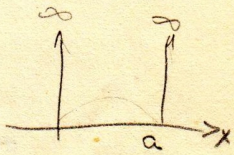
$$= \frac{1}{\sqrt{3\pi}} - \frac{1}{\sqrt{3\pi}}\left[\frac{1}{2}\left(e^{i2\varphi} + e^{-i2\varphi}\right) \cdot e^{-i\frac{2\hbar}{I}t}\right]$$

$$= \frac{1}{\sqrt{3\pi}}\left[1 - \cos 2\varphi \cdot e^{-i\frac{2\hbar}{I}t}\right]$$

II z.

1) Čestica se nalazi u ∞ visokoj potenc. jami širine a i opisana je valnom fjom.

zad 15.



117. str.

$$u(x) = A\, x(a-x)$$

– iz normiranja

$= \sum_{n=0}^{\infty} C_n u_n(x)$ – u Fourier-ov red

$u_n(x) = A_n \sin\frac{n\pi}{a}x$, $\quad A_n = \sqrt{\frac{2}{a}}$, $\quad E_n = \frac{\hbar^2\pi^2}{2ma^2} \cdot n^2$

Izračunati:

a) s kojom se vjerojatnošću čestica nalazi u n-tom energ. nivou

$W_n = ?$ $\quad (W_n = |C_n|^2)$

b) ako je u $t=0$

$\psi(x,0) = A\, x(a-x)$ kako izgleda $\psi(x,t)$ za $t>0$

2) Izračunati srednje vrijednosti

119. str.

$\overline{M^2}$ & $\overline{M_z}$ ako je čestica opisana valnom fjom.

$$u(\vec{r}) = \underset{\text{normir.}}{A}\,(x^2 - y^2 + xy)\, e^{-\frac{x^2+y^2+z^2}{2a^2}}$$

$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$
$z = r\cos\theta$
$x^2 - y^2 = r^2\sin^2\theta(\cos^2\varphi - \sin^2\varphi)$
$xy = r^2\sin^2\theta(\sin\varphi\cos\varphi)$
$x^2 - y^2 + xy = r^2\sin^2\theta\left[\cos 2\varphi + \frac{\sin 2\varphi}{2}\right]$

i naći još M_z & M^2 mogućih vrijednosti

$\left[\Delta + \frac{2m}{\hbar^2}(E - U(x,y,z))\right] u(\vec{r}) = 0$ – naći da li je $U(r)$

ako je: $\hat{H}$, M^2, M_z su dijag. matrice
pa za njih vrijedi: ako $U(r)$
$\hat{G} u_n = g_n u_n \Rightarrow$
$\Rightarrow \bar{G} = \int u^* \hat{G} u \, d\tau$

3) Schröd. jba se može izvesti i varijacionog principa Ritza

$\hat{H} = -\frac{\hbar^2}{2m}\Delta + U = E$

$$E(u) = \frac{\int u^* \hat{H} u \, d\tau}{\int u^* u \, d\tau}$$

funkcional nekе valne fje.

$\hat{\Delta} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\ ) + \frac{1}{r^2}\Lambda$

$-\frac{1}{r^2}\left(\frac{M^2}{\hbar^2}\right) = \hat{L}^2$

- ona fja. koja minimalizira desnu stranu je točno rješ. Schröd. jbe

$\forall$ druga daje veću vrijednost nego što je daje egz. valna fja

→ Naći Ritz-ovom varijacionom metodom energiju osnovnog $\Rightarrow n=1$ $\Rightarrow \rho = \frac{r}{a_0}$ stanja atoma vodika ako se za probnu fju. uzme fja. oblika $U(r) = -\frac{e^2}{r}$ $Z=1$

$\rho = \frac{2Z}{n r_0} r$ $\quad Z=1\ n=1 \Rightarrow \rho = \frac{2}{r_0} r$

$$u(r) = e^{-\frac{\alpha\rho^2}{2}} \qquad \rho = \frac{r}{a_0} \qquad r_0 = a_0 = \frac{\hbar^2}{me^2}$$

egzaktna fja. osnovnog stanja je $e^{-\frac{\rho}{2}}$ jer $u_{00}(r,\vartheta,\varphi) = R_{10}(r)\, Y_0^0(\vartheta,\varphi) = \frac{1}{\sqrt{\pi r_0^3}} e^{-\frac{r}{r_0}}$

$$E(\alpha) = \frac{\int_0^\infty e^{-\alpha\frac{\rho^2}{2}}\left\{\left[-\frac{\hbar^2}{2m}\left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{\hat{L}^2}{r^2}\right) - \frac{e^2}{r}\right] e^{-\frac{\alpha\rho^2}{2}}\right\} 4\pi r^2 dr}{\int_0^\infty e^{-\alpha\rho^2}\cdot 4\pi r^2 dr}$$

probna fja. neovisna o kutu

operator nadjelujе na elem. volumen

zahtijevamo da za taj α E bude minim.

$$\frac{\partial E(\alpha)}{\partial \alpha} = 0 \quad \Rightarrow \alpha_0$$

$$\Rightarrow E_0 = E(\alpha_0)$$ - energija osnovnog stanja

$$u(\vec{r}) = e^{-\alpha\frac{\rho^2}{2}} \qquad \rho = \frac{r}{a_0} \qquad a_0 = \frac{\hbar^2}{4me^2}$$

$$E(\alpha) = \frac{\int u^*(\vec{r})\, H\, u(\vec{r})\, d^3\vec{r}}{\int u^*(\vec{r})\, u(\vec{r})\, d^3\vec{r}}$$

Kutni dio operatora

$$= \frac{\frac{e^2}{2}\int_0^\infty \frac{dr}{a_0^3}\cdot r^2 e^{-\alpha\frac{\rho^2}{2}}\left(-\frac{\frac{1}{a_0^2}\hbar^2}{2m_e}\left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{\vec{M}^2}{\hbar^2 r^2}\right) - \frac{e^2}{r}\frac{1}{a_0}\right)e^{-\frac{\alpha\rho^2}{2}}}{\int_0^\infty e^{-\alpha\rho^2}\cdot r^2\frac{dr}{a_0^3}}$$

$$= -\frac{e^2}{2a_0}\,\frac{\int_0^\infty d\rho\cdot\rho^2 e^{-\frac{\alpha\rho^2}{2}}\cdot\left[\frac{1}{\rho}\frac{d^2}{d\rho^2}\cdot\rho + \frac{2}{\rho}\right]e^{-\frac{\alpha\rho^2}{2}}}{\int_0^\infty e^{-\alpha\rho^2}\cdot\rho^2 d\rho}$$

$$= \underbrace{-\frac{e^2}{2a_0}}_{\text{energija osnovnog stanja}}\,\frac{\int_0^\infty d\rho\cdot\rho e^{-\frac{\alpha\rho^2}{2}}\left[\frac{d^2}{d\rho^2}\cdot\rho + 2\right]e^{-\frac{\alpha\rho^2}{2}}}{\int_0^\infty d\rho\,\rho^2 e^{-\alpha\rho^2}}$$

$$-\rho^2\alpha\, e^{-\frac{\alpha\rho^2}{2}}$$

$$\frac{d^2}{d\rho^2}\,\rho e^{-\frac{\alpha\rho^2}{2}} = e^{-\frac{\alpha\rho^2}{2}}\left(-\alpha\rho - 2\rho\alpha + \alpha^2\rho^3\right)$$

$$E(\alpha) = -\frac{e^2}{2a_0}\cdot\frac{\int_0^\infty d\rho\, e^{-\alpha\rho^2}\left[\alpha^2\rho^4 - 3\alpha\rho^2 + 2\rho\right]}{\int_0^\infty d\rho\,\rho^2 e^{-\alpha\rho^2}}$$

$$t = \alpha\rho^2 \qquad \rho = \alpha^{-\frac{1}{2}}t^{\frac{1}{2}} \qquad \Rightarrow d\rho = \frac{1}{2}\alpha^{-1/2}t^{-1/2}dt$$

$$E(\alpha) = -\frac{e^2}{2a_0}\cdot\frac{\int_0^\infty dt\cdot t^{-1/2}\cdot e^{-t}\left[t^2 - 3t + 2\alpha^{-1/2}t^{1/2}\right]}{\int_0^\infty dt\; t^{-1/2}\cdot\frac{t}{\alpha}e^{-t}}$$

$$-e^{-t}\Big|_0^\infty = 0+1$$

$$= -\frac{e^2}{2a_0}\,\frac{\alpha\left[\int_0^\infty t^{3/2}e^{-t}dt - 3\int_0^\infty t^{1/2}e^{-t}dt + 2\alpha^{-1/2}\int_0^\infty e^{-t}dt\right]}{\int_0^\infty t^{1/2}e^{-t}dt}$$

$$\frac{4!}{2^4\cdot 2!}\sqrt{\pi} = \frac{1\cdot 2\cdot 3\cdot 4}{4\cdot 2\cdot 2}$$

$$\frac{2}{4\cdot 1}\sqrt{\pi}$$

$$= -\frac{e^2}{2a_0}\,\frac{\alpha\left[\Gamma(\frac{5}{2}) - 3\Gamma(\frac{3}{2}) + 2\alpha^{-1/2}\right]}{\Gamma(\frac{3}{2})} = -\frac{e^2}{2a_0}\cdot\frac{\alpha\left[\frac{3}{4}\sqrt{\pi} - 3\frac{\sqrt{\pi}}{2} + 2\alpha^{-\frac{1}{2}}\right]}{\frac{1}{2}\sqrt{\pi}}$$

$$E(\alpha) = -\frac{e^2}{2a_0}\left[\frac{3}{2}\alpha - 3\alpha + \frac{4}{\sqrt{\pi}}\alpha^{1/2}\right]$$

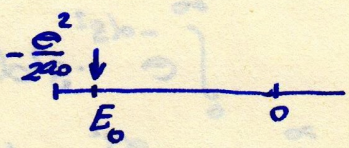
$$= -\frac{e^2}{2a_0}\left[\frac{4}{\sqrt{\pi}}\alpha^{1/2} - \frac{3}{2}\alpha\right]$$

$$0 = \frac{\partial E(\alpha)}{\partial \alpha} = -\frac{e^2}{2a_0}\left[\frac{2}{\sqrt{\pi}}\alpha^{-1/2} - \frac{3}{2}\right]$$

$$\Rightarrow \alpha^{1/2} = \frac{4}{3\sqrt{\pi}}$$

$$E(\alpha_0) = -\frac{e^2}{2a_0}\left[\frac{16}{3\pi} - \frac{8}{3\pi}\right]$$

$$E_0 = -\frac{e^2}{2a_0}\cdot\frac{8}{3\pi}$$



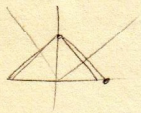
– varijacionom metodom dobijemo višu energiju

$$\Rightarrow u(\vec{r}) = \underset{\text{norm.}}{A}\, e^{-\frac{2}{3\sqrt{\pi}}\overset{\alpha}{\xi}\, r^2} = A\, e^{-\frac{8}{9\pi} r^2}$$

II:

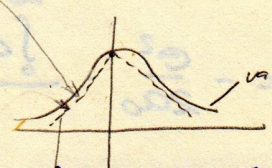
1) Naći energiju osnovnog stanja H.O. ako se za probnu fju. uzme

$$u(x) = e^{-\alpha|x|} \qquad |x| = \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$$



$$-|x|+1 = \begin{cases} -x+1 & x>0 \\ +x+1 & x<0 \end{cases}$$

Pažnja!



$$\frac{d}{dx}|x| = \begin{cases} 1 \\ -1 \end{cases} = \varepsilon(x) = S(x) - S(-x)$$

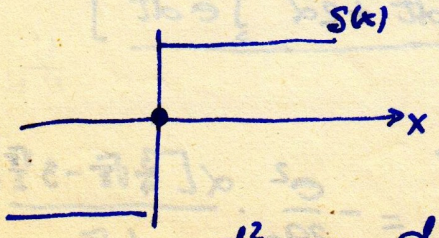
dvostruko stepenasta fja.

derivacija nepr. fja. – ∃ singularitet

Hajsenberg pažljivo tretirati

– integral se ne smije rastaviti na dva dijela, jer se pri tom gubi singularitet iz vida

– tj. integral se ne smije rastaviti na dva dijela (parne fje) jer je singularitet bitan



Najprije da biti Hamiltonijan koji djeluje na valnu fju, pa tek onda integrirati

$$\Rightarrow \frac{d^2}{dx^2}|x| = \frac{d}{dx}\varepsilon(x) = 2\delta(x)$$

$$\frac{d}{dx}S(x) + \frac{d}{d(-x)}S(-x) = \delta(x) + \delta(-x) = 2\delta(x)$$

Kvantna statistička fizika

za fotonski plin

$f(\omega) = -KT \ln Z \qquad Z = \frac{1}{1-e^{-\beta\hbar\omega}} \qquad e^{-\beta\frac{\hbar\omega}{2}}$

$F = 2\sum_{\vec{k}} f(\omega)$

2 polar.

$$F = 2KT \sum_{\vec{k}} \ln\left(1 - e^{\nu - \frac{\hbar\omega_{\vec{k}}}{KT}}\right)_{\nu=0} \qquad \nu = \frac{\mu}{KT}$$

za foton $\omega_{\vec{k}} = c \cdot k$

1) Nađi broj fotona u V tj. N_f, U, C_v, S, P, G

$\sum \to \int$

$$2\sum_{\vec{k}} = \frac{2V}{(2\pi)^3}\int d^3k$$

standardni postupak (opravd. stavlj. period. uvjeta na kutiju)

$$= \frac{V}{\pi^2}\int k^2 dk = \frac{V}{\pi^2 c^3}\int \omega^2 d\omega$$

$$F = \frac{V(KT)}{\pi^2 c^3}\int_0^\infty d\omega\, \omega^2 \ln\left(1 - e^{\nu - \frac{\hbar\omega}{KT}}\right)_{\nu=0}$$

$$x = \frac{\hbar\omega}{KT}$$

$$F = \frac{V(KT)^4}{\pi^2(\hbar c)^3}\int_0^\infty dx\, x^2 \underbrace{\ln\left(1 - e^{\nu - x}\right)}_{-\sum_{n=1}^{\infty}\frac{e^{n(\nu-x)}}{n}}_{\nu=0}$$

$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$ za $-1 \le x < 1$

$\nu = 0 \Rightarrow 0 < e^{-\frac{\hbar\omega}{KT}} < 1$ za $\frac{\hbar\omega}{KT} > 0$

$\Rightarrow e^{-x} < 1$

$nx = u$

$$F = -\frac{V(KT)^4}{\pi^2(\hbar c)^3}\sum_{n=1}^{\infty}\frac{e^{n\nu}}{n}n^{-3}\int_0^\infty dx \cdot \frac{x^2}{n^{-3}}e^{-nx} = \int_0^\infty du\, u^2 e^{-u} = \Gamma(3) = 2$$

$$F = -\frac{2V(KT)^4}{\pi^2(\hbar c)^3}\sum_{n=1}^{\infty}\left(\frac{e^{n\nu}}{n^4}\right)_{\nu=0} = -\frac{2V(KT)^4}{\pi^2(\hbar c)^3}\cdot\zeta(4)$$

$$F = -\frac{2V(kT)^4}{\pi^2(\hbar c)^3}\sum_{n=1}^{\infty}\left(\frac{e^{n\nu}}{n^4}\right)_{\nu=0} \qquad \nu = \frac{\mu}{kT}$$

$$N_f = -\left(\frac{\partial \beta F}{\partial \nu}\right)_{\beta,V} = -\left(\frac{\partial F}{\partial \mu}\right)_{\beta,V} = \frac{2V(kT)^3}{\pi^2(\hbar c)^3}\sum_{n=1}^{\infty}\left(\frac{e^{n\nu}}{n^3}\right)_{\nu=0}$$

sad možemo staviti $\nu=0$ kad smo našli N_f

$$N_f = \frac{2V(kT)^3}{\pi^2(\hbar c)^3}\cdot 1{,}202 \qquad \zeta(3) = \sum_{n=1}^{\infty}\frac{1}{n^3} = 1{,}202$$

Riemann-ova fja.

$$F = -\frac{2V}{\pi^2(\hbar c)^3}\cdot\beta^{-4}\,\zeta(4)$$

$$U = \left(\frac{\partial \beta F}{\partial \beta}\right)_V = \frac{3\cdot 2\,V(kT)^4}{\pi^2(\hbar c)^3}\,\zeta(4) \qquad \zeta(4) = \frac{\pi^4}{90}$$

$$U = \frac{V\cdot\pi^2 k^4}{15(\hbar c)^3}\cdot T^4 = V\cdot a\cdot T^4$$

Stefan-Boltzman

$$a = \frac{\pi^2 k^4}{15(\hbar c)^3} = 7{,}563\cdot 10^{-15}\ \frac{erg}{cm^3\,st^4}$$

$$S = -\left(\frac{\partial F}{\partial T}\right) = \frac{4}{3}VaT^3$$

$$P = -\left(\frac{\partial F}{\partial V}\right) = \frac{\pi^2 k^4 T^4}{45(c\hbar)^3} = \frac{1}{3}\cdot\frac{U}{V}$$ — gustoća energije

vrijedi i za ultrarelativistički plin

$$G = F + PV = 0$$ budući je $G \sim \mu = 0$ za fotonski plin

Dz. 2) Pokazati da unutrašnja energija idealnog Bose-plina ima oblik

$$U = \frac{3}{2}kTV\left(\frac{2\pi m kT}{h^2}\right)^{3/2}\sum_{n=1}^{\infty}\frac{e^{\frac{n\mu}{kT}}}{n^{5/2}}$$

ako je degeneracija slaba

tj. $\mu<0$ nema strogo kez. sist.

Karakteristični izraz $e^{-(\epsilon-\mu)\beta} < 1$ $\quad \mu<\epsilon$, $\mu<0$

pa se po njoj može razviti taj izraz

IIz.

1. Izračunati kem. potencijal i unutrašnju energiju Fermijevog plina

REZ:
$$\mu(T) = \mu_0 \left[1 - \frac{\pi^2}{12} \left(\frac{KT}{\mu_0}\right)^2 + \dots \right]$$

$$U = \underbrace{\frac{3}{5} N \mu_0}_{U_0} \left[1 + \frac{5\pi^2}{12} \left(\frac{KT}{\mu_0}\right)^2 + \dots \right]$$

uputa: $N = \int_0^\infty S(E) \cdot g(E)\, dE$ i primjeni razvoj

za $T=0$ $\quad N = \int_0^{\mu_0} g(E)\, dE$ $\quad$ sa μ_0 ići natrag u prvu formulu

$$N = \int_0^\infty S(E) g(E)\, dE = \gamma \int_0^\infty S(E)\, E^{1/2} dE = \gamma \int_0^\infty S(E) \cdot \frac{\partial \overbrace{\left(\frac{2}{3} E^{3/2}\right)}^{=\varphi(E)}}{\partial E} dE$$

$$\varphi(E) = \frac{2}{3} E^{3/2}$$
$$\varphi^{(1)}(E) = E^{1/2}$$
$$\varphi^{(2)}(E) = \frac{1}{2} E^{-1/2}$$

$$N = \gamma \left\{ \varphi(\mu) + \frac{\pi^2}{6} (KT)^2 \varphi^{(2)}(\mu) + \dots \right\}$$

$-\frac{1}{2} - \frac{3}{2} = -\frac{4}{2} = -2$

$$N = \gamma \left\{ \frac{2}{3} \mu^{3/2} + \frac{\pi^2}{6} (KT)^2 \cdot \frac{1}{2} \cdot \frac{3}{2} \mu^{-1/2} + \dots \right\}$$

$\alpha = -\frac{\mu}{KT}$

$$N = \frac{2}{3} \gamma \mu^{3/2} \left[1 + \frac{\pi^2}{8} \left(\frac{KT}{\mu}\right)^2 + \dots \right]$$

→ samo za kvadratnomčlanu jer $\frac{KT}{\mu} \ll 1$

čak i kod normalnih temp.

za $T=0$ $\quad N = \frac{2}{3} \gamma \mu_0^{3/2}$ $\quad = \int_0^{\mu_0} \gamma E^{1/2} dE = \gamma \cdot \frac{2}{3} \mu^{3/2}$

$$U = \int_0^\infty S(E)\, E g(E)\, dE = \gamma \int_0^\infty S(E) \cdot E^{3/2} dE = \gamma \int_0^\infty S(E) \cdot \frac{\partial \overbrace{\left(\frac{2}{5} E^{5/2}\right)}^{\varphi(E)}}{\partial E} dE$$

$\frac{1}{2} - \frac{5}{2} = -\frac{4}{2} = -2$

$$\varphi(E) = \frac{2}{5} E^{5/2}$$
$$\varphi'(E) = E^{3/2}$$
$$\varphi''(E) = \frac{3}{2} E^{1/2}$$

$$U = \gamma \left[\frac{2}{5} \mu^{5/2} + \frac{\pi^2}{6} (KT)^2 \cdot \frac{3}{2} \cdot \frac{5}{2} \mu^{1/2} + \dots \right]$$

$$U = \frac{2}{5} \gamma \mu^{5/2} \left[1 + \frac{5\pi^2}{8} \left(\frac{KT}{\mu}\right)^2 + \dots \right]$$

za $T=0$ $\quad U_0 = \frac{2}{5} \gamma \mu_0^{5/2} = \frac{2}{5} \cancel{\gamma} \mu_0 \frac{3}{2} \frac{1}{\cancel{\gamma}} N = \frac{3}{5} N \mu_0$

za idealni plin $\quad U = \frac{3}{2} P \cdot V$

$$\Rightarrow P_0 = \frac{2}{3} \cdot \frac{U_0}{V} = \frac{2}{3V} \cdot \frac{2}{5} \gamma \cdot \mu_0^{5/2}$$

str. 61

$$\mu_0^{3/2} = \frac{N}{\frac{2}{3}\gamma} = \mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2\right] \qquad \frac{kT}{\mu} \ll 1$$

pa zadovoljeno i $\frac{kT}{\mu} < 1$

$$\mu_0^{3/2} = \mu^{3/2} + \frac{\pi^2}{8}\left(\frac{kT}{\mu}\right)^2 \mu^{3/2}$$

$$\mu_0^{3/2} = \mu^{3/2} + \frac{\pi^2}{8}(kT)^2 \cdot \mu^{-1/2} \quad \Big/ \frac{d}{dT}$$

$$0 = \frac{3}{2}\mu^{1/2}\cdot\mu'(T) - \frac{1}{2}\cdot\frac{\pi^2}{8}(kT)^2\mu^{-3/2}\cdot\mu'(T) + \frac{2\pi^2}{8}k^2 T\,\mu^{-1/2} \quad \Big/ \cdot \mu^{-1/2}$$

$$0 = \frac{3}{2}\mu'(T) - \frac{\pi^2}{16}\left(\frac{kT}{\mu}\right)^2\cdot\mu'(T) + \frac{\pi^2}{4}k^2\frac{T}{\mu} \quad \Big/ \cdot 2$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots \quad \text{za } |z| < 1$$

$$\Rightarrow \quad \mu'(T) = \frac{-\frac{\pi^2}{2}k^2\frac{T}{\mu}}{\underset{1}{3} - \underset{24}{\frac{\pi^2}{8}}\left(\frac{kT}{\mu}\right)^2} \qquad z = \frac{\pi^2}{24}\left(\frac{kT}{\mu}\right)^2 \ll 1$$

$$= -\frac{\pi^2}{6}k\left(\frac{kT}{\mu}\right)\left[1 + \frac{\pi^2}{24}\left(\frac{kT}{\mu}\right)^2 + \frac{\pi^4}{24^2}\left(\frac{kT}{\mu}\right)^4 + \dots\right]$$

$$\left(\frac{kT}{\mu}\right)^3 \to 0$$

$$\mu'(T) \simeq -\frac{\pi^2}{6}k^2\frac{T}{\mu(T)} \xrightarrow{T\to 0} 0$$

$$\mu''(T) \simeq -\frac{\pi^2}{6}k^2\cdot\frac{1}{\mu(T)} + \frac{\pi^2}{6}k^2 T\,\frac{1}{\mu^2(T)}\cdot\mu'(T) \equiv -\frac{\pi^2}{6}k^2\frac{1}{\mu(T)} + \frac{\pi^2}{6}k^2 T\,\frac{1}{\mu^2(T)}\cdot\left(-\frac{\pi^2}{6}k^2\frac{T}{\mu(T)}\right) = -\frac{\pi^2}{6}k^2\frac{1}{\mu(T)} - \frac{\pi^4}{36}k^4 T^2\,\frac{1}{\mu^3(T)}$$

$\to 0$ za $T \to 0$

$$\Rightarrow \quad \mu''(T) \simeq -\frac{\pi^2}{6}k^2\cdot\frac{1}{\mu(T)} \xrightarrow{T\to 0} -\frac{\pi^2}{6}k^2\frac{1}{\mu_0}$$

$$\mu(T) = \mu(0) + \frac{T}{1!}\mu'(0) + \frac{T^2}{2!}\mu''(0) + \dots$$

Kasnije se vidi da viši članovi u potencijama od $\left(\frac{kT}{\mu_0}\right)$ pa otpadaju

$$\mu(T) = \mu_0 - \frac{\pi^2}{12}(kT)^2\cdot\frac{1}{\mu_0} + \dots$$

$$\Rightarrow \quad \mu(T) = \mu_0\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2 + \dots\right]$$

$$\mu \gg kT \quad \Rightarrow \quad \mu_0 \gg kT$$

$$U = \frac{2}{5}\gamma\, \mu_0^{5/2}\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2\right]^{5/2}\cdot\left\{1+\frac{5\pi^2}{8}\left(\frac{kT}{\mu_0}\right)^2\cdot\left[1-\frac{\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2\right]^{-2}\right\}$$

$$\left[1-\frac{5}{2}\left(\frac{\pi^2}{12}\right)\left(\frac{kT}{\mu_0}\right)^2+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\cdot\frac{\pi^4}{144}\cancel{\left(\frac{kT}{\mu_0}\right)^4}-\dots\right] \qquad \left[1+2\cdot\frac{\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2+\cancel{\frac{2\cdot 3}{2!}\frac{\pi^4}{144}\left(\frac{kT}{\mu_0}\right)^4}+\dots\right]$$

$$U = \frac{2}{5}\gamma\, \mu_0^{5/2}\left[1-\frac{5\pi^2}{24}\left(\frac{kT}{\mu_0}\right)^2\right]\left[1+\frac{5\pi^2}{8}\left(\frac{kT}{\mu_0}\right)^2\right]$$

$$1+\left(\underbrace{-\frac{5\pi^2}{24}+\frac{5\pi^2}{8}}_{\frac{10}{24}}\right)\left(\frac{kT}{\mu_0}\right)^2+(\dots)\cancel{\left(\frac{kT}{\mu_0}\right)^4}$$

$$U = \frac{2}{5}\gamma\, \mu_0^{5/2}\left[1+\frac{5\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2\right]$$

$$U = \frac{3}{5}N\mu_0\left[1+\frac{5\pi^2}{12}\left(\frac{kT}{\mu_0}\right)^2\right]$$

Rješ. str. 20.

2) $U = \sum_i E_i N_i = \sum_i g_i \varepsilon_i N_i \rightarrow \int_0^\infty \varepsilon(E)\, E\, g(E)\, dE \qquad g(E) = \underbrace{\frac{(2S+1)V(2m)^{3/2}}{4\pi^2\hbar^3}}_{\gamma}\sqrt{E}$

ako je degeneracija slaba tj. N_0 - broj bozona na nivou $E=0$ mali:

tj. $N_p = N\left(\frac{T}{T_c}\right)^{3/2} \approx N \quad \Rightarrow T \approx T_c$

možemo prijeći $\Sigma \rightarrow \int$ za U, a ne pisati $U \simeq N_p \cdot kT$

$$U = \gamma\int_0^\infty E^{3/2}\varepsilon(E)\,dE = \gamma\int_0^\infty \frac{E^{3/2}}{e^{\frac{E-\mu}{kT}}-1}\,dE \qquad t=\frac{E}{kT} \qquad \nu=\frac{\mu}{kT}$$

$$E^{3/2} = (kT)^{3/2}t^{3/2} \qquad dE = (kT)\,dt$$

$$U = \gamma\cdot(kT)^{5/2}\cdot\int_0^\infty \frac{t^{3/2}}{e^{t-\nu}-1}\,dt$$

fizikti smisao za bozone $\varepsilon_i > 0$

$\Rightarrow \underline{E-\mu > 0} \qquad E > \mu$

$\Rightarrow t-\nu > 0 \Rightarrow e^{t-\nu} > 1$

$\Rightarrow \underline{e^{\nu-t} < 1}$

$$e^{\nu-t}\cdot\frac{1}{1-e^{\nu-t}} = e^{\nu-t}\sum_{n=0}^{\infty} e^{n(\nu-t)}$$

$$U = \gamma\cdot(kT)^{5/2}\sum_{n=1}^{\infty} e^{n\nu}\cdot\int_0^\infty t^{3/2}\cdot e^{-nt}\,dt$$

$nt = u \qquad dt = \frac{1}{n}du$

$\frac{1}{n^{3/2}}\cdot\frac{1}{n^{1}}$

$$U = \gamma(kT)^{5/2}\cdot\sum_{n=1}^{\infty}\frac{e^{n\nu}}{n^{5/2}}\underbrace{\int_0^\infty u^{3/2}\cdot e^{-u}\,du}_{\Gamma\left(\frac{5}{2}\right)=\frac{4!}{2^4\cdot 2!}\sqrt{\pi}=\frac{\cancel{3}\cdot\cancel{4}}{\cancel{16}_{4}\cdot\cancel{2}}\sqrt{\pi}=\frac{3\sqrt{\pi}}{4}}$$

$$U = \frac{3}{2}(kT)\cdot\gamma\cdot(kT)^{3/2}\cdot\frac{\sqrt{\pi}}{2}\sum_{n=1}^{\infty}\frac{e^{n\nu}}{n^{5/2}}$$

$$\gamma(kT)^{3/2}\frac{\sqrt{\pi}}{2} = \frac{(2S+1)V(2m)^{3/2}(kT)^{3/2}\cdot\pi^{3/2}}{4\pi^2\cdot 2\sqrt{\pi}\,\hbar^3} = \frac{(2S+1)V(2\pi mkT)^{3/2}}{h^3}$$

h^3